Connectivity in Image Processing and Analysis: Theory, Multiscale Extensions and Applications

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To the memory of my mother,
Consuêlo

Contents

Preface

Connectivity plays an important role in image processing and analysis, and particularly in problems related to image segmentation, image filtering, image coding, motion analysis, multiscale signal decomposition, pattern recognition, and other application areas. In this dissertation, we study a general theory of connectivity in image processing and analysis.

Connectivity is classically defined using either a topological or a graph-theoretic framework, and their fuzzy analogs. We provide a thorough review of several existing definitions of connectivity. Although these classical concepts have been extensively applied in image processing and analysis, they are unfortunately incompatible. The theory of connectivity classes, first proposed in the late eighties for binary images, and recently extended to arbitrary complete lattices, circumvents the shortcomings of classical definitions by providing a consistent unified theoretical framework that includes the majority of the existing concepts of connectivity. We review this theory, expand it with new results and examples, and demonstrate its usefulness in applications based on connected operators.

We also propose the notion of multiscale connectivity. We provide a novel theoretical framework for multiscale connectivity, which includes the theory of connectivity classes in complete lattices as a special, single-scale case. Among the items we propose and study in connection with multiscale connectivities is the integration of connectivity with multiscale methods that are currently routinely employed in image processing and analysis applications. In particular, we define several multiscale tools based on multiscale connectivities, such as multiscale signal decompositions, hierarchical segmentation, hierarchical clustering and multiscale features. Several examples of application of these multiscale tools are provided using synthetic and real images.

Many people contributed to make this work possible. In what follows, I mention just a few of them. The help of all those who should have been mentioned here but were left out is gratefully acknowledged as well.

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Chapter 1

Introduction

Connectivity plays an important role in image processing and analysis, and particularly in problems related to image segmentation, image filtering, image coding, motion analysis, multiscale signal decomposition, pattern recognition, and other application areas. In this dissertation, we study a general theory of connectivity in image processing and analysis. We review the existing theory of connectivity, expand it with new results and examples, and demonstrate its usefulness in applications. We also propose a novel theoretical framework for multiscale connectivity, which includes most of the previous notions of connectivity as special cases. Among the contributions the theory of multiscale connectivity makes is the possibility of integrating connectivity with multiscale methods that are currently routinely employed in image processing and analysis applications. In the next few sections, we outline the content of this dissertation.

1.1 Background

In mathematics, and in image processing and analysis in particular, connectivity is classically defined using either a *topological* or a *graph-theoretic* framework [22, 60]. More recently, these classical notions have been extended to a *fuzzy* setting [18, 67, 95, 96], which allows the definition of connectivity for grayscale images, as well.

Although these classical concepts have been extensively applied in image processing and analysis, they are incompatible [66]. In general, topological connectivity is useful for images defined over a continuous space, whereas graph-theoretic connectivity is useful for images defined over a discrete space. Compatibility is desired, since discrete images are often obtained from discretization of continuous scenes, by means of sampling. In addition to incompatibilities between the classical approaches to connectivity, there are conceptual limitations as well. A topological or a graph-theoretic framework to connectivity limits the type of objects to which connectivity can be applied.

This state of affairs motivated G. Matheron and J. Serra to propose an axiomatic approach to connectivity, known as the theory of *connectivity classes*, which circumvents the shortcomings of the classical definitions [77]. This approach is based on the observation that standard notions of connectivity share the properties that the empty set and the points in the space are connected, and that unions of intersecting connected objects are connected. This may be considered to be a minimal set of desirable requirements for connectivity. In particular, they imply that an object is *partitioned* by its connected components.

Matheron and Serra's original theory was set-oriented, and therefore could be applied only to binary images. It turns out that there exists a very successful theoretical framework for studying operations on binary, grayscale, and multispectral images, as well as on more general objects. This framework is based on mathematical entities known as *complete lattices* [8]. For example, the family of all binary (resp. grayscale, multispectral) images, provided with a meaningful *partial order relation*, is an example of a complete lattice. Recently, Serra showed how to extend the theory of connectivity classes to complete lattices, in a way that is consistent with the binary theory [78–80]. This framework includes and unifies traditional concepts of connectivity and allows the study of many interesting connectivity examples that are not covered by the classical definitions. We remark that the field that studies operators on complete lattices is commonly known as *mathematical morphology* [34, 56, 76]. The term derives from the fact that the prototypical morphological operators for binary (resp. grayscale, multispectral) images have a clear geometrical meaning. Since it is based on complete lattices, the theory of connectivity classes can be called a theory of "morphological connectivity."

On the other hand, multiscale methods have proven to be very important in image processing and analysis. These methods include linear pyramid decomposition techniques [15], scale-space methods [38, 91], hierarchical clustering algorithms [36, 84], hierarchical segmentation methods [16, 47], wavelet representations [20, 53], multiscale classification features [55], and other areas of application. All these methods share a common unifying property: they propose to analyze an image at several different *scales*. The issue of scale arises in image processing and analysis for several reasons. For instance, a complete description of an image can be obtained by combining global information at small scales with detail information at large scales (we remark that the term scale is used here in the sense of *resolution*, which is the inverse of the sense in which it is used, for example, in map making or in scale-space theory). In addition, there is some psychovisual evidence that the human visual system operates in a multiscale fashion [32, 53]. Furthermore, in the real world, measurements and operations make sense only at the appropriate scale.

1.2 Contributions

The contributions made in this dissertation can be grouped into two broad categories. First, we enrich the existing theory of connectivity classes in complete lattices, by providing several new theoretical results and introducing new examples of connectivity. Moreover, we provide some examples of practical image processing and analysis applications using a class of operators based on connectivity criteria. Second, we propose a novel theoretical framework for connectivity in a multiscale setting, which includes the previous theory as a special, single-scale case, and we demonstrate its application in a few multiscale image processing and analysis tasks.

1.2.1 Connectivity on Complete Lattices

As mentioned before, connectivity is classically defined using a topological or graph-theoretic framework, and their fuzzy analogs. In this dissertation, we provide an extensive review of several existing definitions of connectivity. In particular, we show that some of the examples of fuzzy connectivity that we examine lead to connectivity classes.

An important and useful aspect of the theory of connectivity classes is that a connectivity can be equivalently specified by either a family of connected objects (the connectivity class) or by a family of operators known as *connectivity openings*, which perform connected component extraction. This fact has been recognized from the beginning, in the binary case [77], and later extended to the complete lattice case in [78]. In this dissertation, we provide new results on semi-continuity properties of connectivity openings.

Another important operator is the *reconstruction operator*. This operator has been known in mathematical morphology [44, 45] before the introduction of the theory of connectivity classes. In [35], H. Heijmans showed that, in the binary case, a connectivity can be equivalently specified in terms of a reconstruction operator. In this dissertation, we extend

Heijmans' binary result to *infinite* $\vee$*-distributive* complete lattices; this cover the cases of binary, grayscale and multispectral images (a similar result was independently established in [64]).

Developing meaningful examples of connectivity classes for grayscale and multispectral images has turned out to be a non-trivial problem. In [78–80], J. Serra introduced examples of grayscale connectivity classes in suitably defined complete lattices. In this dissertation, we present a method to construct complete lattices, called ψ*-invariant lattices*, which contain the invariant elements with respect to an appropriately chosen operator ψ. These complete lattices allow us to develop new interesting examples of connectivity classes; these include the classical notion of graph-theoretic k-connectivity, which is not compatible with the original set-oriented definition of connectivity classes, and a novel example of connectivity for grayscale images, called *flat grayscale connectivity*. We study flat grayscale connectivity in a general topological framework, which can be easily specialized to the discrete case. We demonstrate, by means of examples, that flat grayscale connectivity produces meaningful and potentially useful segmentation results.

A *second-generation connectivity class* is a new connectivity class generated from an existing one by means of a suitably defined operator [77–80]. There are basically two categories of second-generation connectivities. The first one is based on clustering elements of the original connectivity by means of a *clustering* operator. We give an axiomatic formulation of clustering operators, and present a new example of a clustering connectivity class based on morphological sampling operators. The second category is the dual, in a sense, of the first. It is based on restricting a given connectivity class by means of a *contraction* operator. This includes the known case of a connectivity class restricted by openings, previously studied for the binary case in [65, 66], which we generalize to *atomic* complete lattices.

An alternative approach to the theory of connectivity classes is the concept of hyperconnectivity, proposed by J. Serra in [78, 79]. The hyperconnectivity approach modifies one of the axioms required by the theory of connectivity classes, thereby allowing new examples of connectivity, including interesting grayscale examples. The main drawback of this approach is that it loses much of the structure and strength of the theory of connectivity classes. In this dissertation, we study in detail this alternative approach to connectivity. We give new examples, including a few classical definitions of connectivity that fit naturally in the hyperconnectivity framework. In addition, we define a new class of operators that are useful for segmentation, called Z*-operators*, and propose a novel segmentation technique for grayscale and multispectral images, which we call the *segmentation by similarity zones*.

Finally, we study a class of operators, known as *connected operators* [19, 35, 74], which are defined in terms of a given connectivity class. Connected operators do not work at the pixel level, but rather at the level of the *flat zones* of an image. A connected operator can remove boundaries, but cannot shift boundaries or introduce new ones. It therefore preserves contour/shape information, known to carry most of image content perceived by human observers. In this dissertation, we study connected operators in some detail. The main contribution here is a number of applications, taken from our previous work, which demonstrate the effectiveness of connected operators in various image processing and analysis tasks, including landmine detection in multispectral images [9], target detection and tracking in FLIR video sequences [10], and topology correction of 3-D brain MRI data (this last work has not been published in its original form; however, it led to the development of a related method that was reported in [30, 31]).

1.2.2 Multiscale Connectivity

In this dissertation, we present a novel axiomatic framework for the notion of *multiscale connectivity*. As in the case of connectivity classes, the theory of multiscale connectivity is based on complete lattices. The proposed framework includes the previous theory of connectivity classes as a special case; a connectivity class can be seen as a single-scale connectivity. Hence, the proposed theory can be said to be a theory of "multiscale morphological connectivity."

The idea of multiscale connectivity arises naturally from the observation that the connectivity of an object depends upon the scale at which it is observed. This means that an object may be assigned a varying degree of connectivity, and that several levels of connectivity of varying strictness can be defined. The first observation leads to the notion of a *connectivity measure*, whereas the second leads to the notion of a *connectivity pyramid*. In this dissertation, we provide an axiomatic formulation of these concepts and prove that they are equivalent. We show that the multiscale analog of classical topological (resp. graph-theoretic) connectivity is given by fuzzy topological (resp. graph-theoretic) τ*-connectivity*, which corresponds to connectivity at scale τ. We define and investigate the notions of σ-connectivity openings and σ-reconstruction operators associated with a multiscale connectivity, which correspond to connectivity openings and reconstruction operators at scale σ, respectively. We distinguish between two cases of interest. The first case, to be referred to as *continuous multiscale connectivity*, assumes that the scale parameter is continuous. The second case, to be referred to as *discrete multiscale connectivity*, assumes that the set of scales available for multiscale analysis is discrete, as is usually the case in digital image processing and analysis. We show that, in a precise sense, discrete multiscale connectivity is a special case of continuous multiscale connectivity.

In Mathematical Morphology, there are several examples of operators that have a natural multiscale interpretation. We define and study examples of multiscale connectivities generated by such operators. We consider two general categories of multiscale operators, namely, clustering pyramids, which lead to "negative" multiscale connectivities, and contraction pyramids, which lead to "positive" multiscale connectivities.

We also investigate the problem of creating new multiscale connectivities from existing ones by means of a suitably defined operator. Following J. Serra's nomenclature for the single-scale connectivity case [78], we refer to these new multiscale connectivities as *second-generation multiscale connectivities.*

In addition, we show how the proposed framework leads to the development of useful multiscale image processing and analysis tools, such as pyramid decomposition, hierarchical segmentation, hierarchical clustering, and multiscale features. These are described next.

- **Pyramid Decomposition.** Discrete multiscale connectivities lead to an interesting example of a nonlinear multiscale signal decomposition scheme, which uses σ-reconstruction operators as the analysis operators of a nonlinear pyramid decomposition scheme. In contrast to pyramid decompositions based on pixel-based operators [27], the pyramid decomposition that we propose here does not work at the pixel level, but at the level of the connected components of an object at various scales. This leads to a novel *object-based* multiscale signal decomposition scheme. This scheme can be considered to be the pyramid transform analog of the so-called *second-generation* image coding techniques [40], which constitute an object-based approach to image compression that codes homogeneous regions (objects) in an image.

- **Hierarchical Segmentation and Hierarchical Clustering.** The concept of hierarchical segmentation is of fundamental importance in multiscale applications, such as adaptive bit-rate object-based coding of still images and image sequences [72]. In these applications, it is desirable to have several levels of segmentation at various scales, so that the amount of compression (bit-rate) can be adjusted to meet varying transmission/storage requirements. On the other hand, hierarchical clustering is a technique used to group together similar objects in a hierarchical fashion, with applications in unsupervised classification algorithms, where the number, or the statistical distribution, of classes is not known a priori [23, 36]. The availability of several levels of clustering in hierarchical clustering algorithms is often helpful in revealing the true structure of the data; e.g., the number of classes that best represent the organization of the data. Both these notions can be formalized in the context of *hierarchical partitions*. In this dissertation, we propose two examples of hierarchical partitions, based on multiscale connectivities, which can be used for hierarchical segmentation and hierarchical clustering algorithms: the *hierarchical partition of connected components*, which can be defined in arbitrary complete lattices, and the *hierarchical partition of flat zones*, which applies to binary, grayscale, and multispectral images.

- **Multiscale Features.** Image features are fundamental constituents of pattern recognition algorithms for image analysis. The performance of such algorithms is directly related to the choice of robust features. In this dissertation, we propose two multiscale image analysis features, namely, the *clustering curve* and the *clustering spectrum*, which measure the multiscale connectivity properties of a given object. We remark that, in mathematical morphology, a very useful and well-known example of multiscale image analysis feature is the *pattern spectrum* [55]. The clustering spectrum is distinct from the pattern spectrum, since the latter is based on measurements made on a granulometric distribution, whereas the former is based on measurements made on a hierarchical partition. Nevertheless, clustering spectra and pattern spectra are similar tools and share similar properties.

Several examples, using synthetic and real images, are employed to illustrate the use of the multiscale tools described above.

We also investigate the notion of multiscale hyperconnectivity. We show that the classical notion of graph-theoretic degree of connectivity is an example of a hyperconnectivity measure. We also construct a meaningful example of grayscale multiscale hyperconnectivity, which we call multiscale flat hyperconnectivity; this is the multiscale extension of the original example of hyperconnectivity proposed in [78]. We present an example of a multiscale signal decomposition scheme based on multiscale flat hyperconnectivity.

Finally, we show that connected operators can be extended to the framework of multiscale connectivity. We define and study σ-connected operators, which are operators that are connected at scale σ. We show that this leads to a degree of connectivity for connected operators, which measures how "insensitive" the operator is to clustering of flat zones of an image at low connectivity scales. We also define and study *σ-grain operators*. In particular, we show that families of σ-grain openings and σ-grain closings constitute *granulometries* and *anti-granulometries*, respectively, parameterized by the connectivity scale.

An early version of the theory of multiscale connectivity appeared in [13]. However, there are some differences between the approach developed in that paper and the one adopted here. The main differences are that, in this dissertation, the degree of connectivity is allowed to be negative, and the set of scales is fixed, being either the set of real numbers, in the case of continuous scale, or the set of integers, in the case of discrete scale.

1.3 Organization

This dissertation is organized as follows.

In Chapter 2, we introduce our notation and briefly review basic mathematical concepts that will be needed in the sequel. We review key notions of complete lattices, morphological operators, topological spaces, the hit-or-miss topology, and fuzzy sets and fuzzy topological spaces.

In Chapter 3, we provide a thorough review of several classical notions of connectivity on topological spaces and graphs, both in the ordinary and fuzzy sense. We examine the classical notions of connectivity and path-connectivity in topological spaces, and connectivity and k-connectivity in graphs. Several definitions of connectivity on fuzzy topological spaces have appeared in the literature. We examine two of them, namely the notion of fuzzy topological τ-connectivity and *level connectivity*, which are both natural extensions of the notion of ordinary topological connectivity. We then define the notion of fuzzy graphs and investigate the notions of fuzzy graph-theoretic τ-connectivity and *topographic connectivity*.

In Chapter 4, we present the theory of connectivity classes, including our new results and examples concerning connectivity openings, the reconstruction operator, connectivity on ψ-invariant lattices, second-generation connectivity and hyperconnectivity. The material in Chapters 2, 3 and 4 correspond, with a few minor modifications, to what was reported in [12]. Some of this material also appeared in [14].

In Chapter 5, we investigate the notion of connected operators, and demonstrate their effectiveness in various image processing and analysis tasks, by using applications taken from our previous work.

In Chapter 6, we present our theory of multiscale connectivity. We examine continuous and discrete multiscale connectivity, multiscale connectivity examples generated by multiscale morphological operators, and second-generation multiscale connectivity. Moreover, we demonstrate the application of multiscale connectivity in important multiscale image processing and analysis tasks. We also study multiscale hyperconnectivity and the notion of multiscale connected operators. The material in this chapter corresponds what was reported in [11].

Finally, Chapter 7 contains concluding remarks and directions for future research.

We remark here that the examples presented in this dissertation were coded in MatLab 5.3 [57], with the MMach Mathematical Morphology toolbox [58]. Many of the images used in the examples accompany the MMach toolbox and are used under permission from SDC Information Systems.

Chapter 2

Mathematical Preliminaries

The purpose of this chapter is to introduce our notation and to briefly review basic mathematical concepts that will be needed later. Namely, we review key notions of complete lattices, morphological operators, topological spaces, the hit-or-miss topology, and fuzzy sets and fuzzy topological spaces. For a more detailed exposition on those subjects, the reader is referred to [8, 18, 24, 34, 56, 59, 60, 76, 96].

2.1 Complete Lattices

A *partially ordered set* $(\mathcal{L}, \leq)$, also known as a *poset*, is a nonempty set $\mathcal{L}$ together with a binary order relation $\leq$ on $\mathcal{L}$ that is reflexive, anti-symmetric and transitive [8, 34]. A poset $(\mathcal{L}, \leq)$ is said to be a *complete lattice* if every family $\mathcal{K} \subseteq \mathcal{L}$ has an infimum and a supremum in $\mathcal{L}$ [8, 34], denoted by $\bigwedge \mathcal{K}$ and $\bigvee \mathcal{K}$, respectively. A poset in which there is an infimum (resp. supremum) operation is said to be a complete inf (resp. sup) *semi-lattice.* A complete lattice $(\mathcal{L}, \leq)$ for which the order relation is a total order (i.e., for which $A, B \in \mathcal{L}$ implies that $A \leq B$ or $B \leq A$) is called a *complete chain.* Following [79], whenever we use the terms "lattice" and "chain" we mean "complete lattice" and "complete chain," respectively. In addition, we often refer to "lattice $\mathcal{L}$," when there is no confusion as to the underlying partial order.

The prototypical example of lattice is the collection $\mathcal{L} = \mathcal{P}(E)$ of all subsets of a set E, with set inclusion as the partial order; the infimum and supremum are given by set intersection and set union, respectively. The prototypical examples of chains are given by the "completed" set of reals $\overline{\mathbb{R}} = \mathbb{R} \cup \{-\infty, \infty\}$ and the "completed" set of integers

$\overline{\mathbb{Z}} = \mathbb{Z} \cup \{-\infty, \infty\}$, with the usual numerical ordering as the partial order; the infimum and supremum are given by the usual numerical infimum and supremum.

By definition, every lattice $\mathcal{L}$ must possess a *least element* O and a *greatest element* I, given by $O = \bigwedge \mathcal{L}$ and $I = \bigvee \mathcal{L}$, respectively. For example, if $\mathcal{L} = \mathcal{P}(E)$, then $O = \emptyset$ and $I = E$. In a lattice, every element is both an upper bound and a lower bound of the empty set; therefore, $\bigvee \emptyset = O$ and $\bigwedge \emptyset = I$.

The following result appears in [34, Prop. 2.12].

2.1.1 Proposition. Given a poset $\mathcal{L}$, the following three statements are equivalent:

(a) The poset $\mathcal{L}$ is a complete lattice.

(b) The poset $\mathcal{L}$ is a complete inf semi-lattice with a greatest element I.

(c) The poset $\mathcal{L}$ is a complete sup semi-lattice with a smallest element O. □

A subset $\mathcal{S}$ of a lattice $\mathcal{L}$ is called a *sup-generating family* for $\mathcal{L}$ if every element of $\mathcal{L}$ can be written as the supremum of elements in $\mathcal{S}$. For example, the set of points in E is a sup-generating family for the lattice $\mathcal{P}(E)$. An element of the sup-generating family $\mathcal{S}$ is called a *sup-generator*. It is assumed here that O is not a sup-generator; i.e., $O \notin \mathcal{S}$. However, every family $\mathcal{S}$ sup-generates O by means of $O = \bigvee \emptyset$.

Subsets of $\mathcal{L}$ will be denoted by script letters, such as $\mathcal{C}, \mathcal{F}, \mathcal{G}, \mathcal{H}$. The elements of $\mathcal{L}$ are generally denoted by uppercase letters, such as A, B, C; however, in the case of function lattices (see Examples 2.1.2(d)–(f) below), we prefer to use the classical symbols f, g, h. In order to distinguish the elements of the sup-generating family $\mathcal{S}$, we denote them by lowercase letters, such as x, y, z, except in the case of function lattices, where we use the well established "delta" notation.

Given a lattice $\mathcal{L}$, each element $A \in \mathcal{L}$ determines two families in $\mathcal{L}$: the *majorants* of A, given by $\mathcal{M}^*(A) = \{B \in \mathcal{L} \mid B \geq A\}$, and the *minorants* of A, given by $\mathcal{M}_*(A) = \{B \in \mathcal{L} \mid B \leq A\}$. Given a sup-generating family $\mathcal{S} \subseteq \mathcal{L}$ and $A \in \mathcal{L}$, we also define the family $\mathcal{S}(A) = \{x \in \mathcal{S} \mid x \leq A\}$, as the family of all sup-generators majorated by A. Note that $\mathcal{S}(A) = \mathcal{S} \cap \mathcal{M}_*(A)$. Moreover, it is clear that

$$A = \bigvee \{x \mid x \in \mathcal{S}(A)\} = \bigvee \{x \in \mathcal{S} \mid x \leq A\}; \tag{2.1}$$

that is, A is the supremum of the elements of $\mathcal{S}$ that it majorates.

Two lattices $\mathcal{L}$ and $\mathcal{L}'$ are said to be *isomorphic* if there exists a bijection ψ: $\mathcal{L} \to \mathcal{L}'$ that preserves ordering; i.e., for $A, B \in \mathcal{L}$,

$$A \leq B \;\; \Leftrightarrow \;\; \psi(A) \leq \psi(B). \tag{2.2}$$

The bijection ψ is said to be an *isomorphism* between lattices $\mathcal{L}$ and $\mathcal{L}'$. Isomorphic lattices are essentially the same, since they enjoy the same lattice-theoretic properties.

An *atom* is a nonzero element $A \in \mathcal{L}$ (i.e., $A \neq O$) such that $B \leq A$ implies $B = A$ or $B = O$. A lattice $\mathcal{L}$ is said to be *atomic* if there is a sup-generating family of atoms $\mathcal{S}$ in $\mathcal{L}$. For example, the lattice $\mathcal{P}(E)$ is atomic, since the points in E are atoms. A *semi-atom* is a nonzero element $A \in \mathcal{L}$ such that $A \leq A_1 \vee A_2$ implies $A \leq A_1$ or $A \leq A_2$. By induction, this property applies to the supremum of a finite number of elements; i.e., $A \leq A_1 \vee A_2 \vee \cdots \vee A_n$ implies $A \leq A_i$, for some $1 \leq i \leq n$. A semi-atom A is said to be *strong* if the semi-atomicity property works for arbitrary suprema; i.e., $A \leq \bigvee A_\alpha$ implies $A \leq A_{\alpha'}$, for a particular index α'. A lattice $\mathcal{L}$ is said to be (strongly) *semi-atomic* if there is a sup-generating family of (strong) semi-atoms $\mathcal{S}$ in $\mathcal{L}$. For example, the lattice $\mathcal{P}(E)$ is strongly semi-atomic, since the points in E are strong semi-atoms. It is easy to see that all atoms and strong semi-atoms of a lattice $\mathcal{L}$ must be contained in any sup-generating family $\mathcal{S}$ of $\mathcal{L}$. In particular, if $\mathcal{L}$ is sup-generated by a family that does not contain atoms (resp. strong semi-atoms), $\mathcal{L}$ cannot be atomic (resp. strongly semi-atomic). In this dissertation, whenever an atomic, semi-atomic or strongly semi-atomic lattice $\mathcal{L}$ is considered, we implicitly assume a sup-generating family $\mathcal{S}$ that contains only atoms, semi-atoms or strong semi-atoms, respectively.

A lattice $\mathcal{L}$ is said to be *infinite* $\vee$*-distributive* if

$$A \wedge \bigvee B_\alpha \; = \; \bigvee (A \wedge B_\alpha), \tag{2.3}$$

whereas $\mathcal{L}$ is said to be *infinite* $\wedge$*-distributive* if

$$A \vee \bigwedge B_\alpha \; = \; \bigwedge (A \vee B_\alpha), \tag{2.4}$$

for every $A \in \mathcal{L}$ and $\{B_\alpha\}$ in $\mathcal{L}$. A lattice $\mathcal{L}$ is said to be *infinite distributive* if it is both infinite $\vee$-distributive and infinite $\wedge$-distributive. For example, the lattice $\mathcal{P}(E)$ is infinite distributive.

In the following, we give a few examples of lattices and associated sup-generating families, and describe some of their properties.

2.1.2 Example.

(a) (*Set lattice*). As mentioned before, the collection $\mathcal{P}(E)$ of all subsets of a set E is a lattice. The partial order is set inclusion, whereas the supremum and infimum are set union and set intersection, respectively. Lattice $\mathcal{P}(E)$ is sup-generated by $\mathcal{S} = \{\{v\} \mid v \in E\}$. It is atomic, strongly semi-atomic, and infinite distributive.

(b) (*Lattice of closed sets*). Let E be a topological space (for a brief overview of topological concepts, see Section 2.3). The collection $\mathcal{F}(E)$ of all closed subsets of E is a lattice. The partial order is set inclusion, the infimum is set intersection, whereas the supremum is the topological closure of set union. Now, assume that E is a Hausdorff space. Then, $\mathcal{F}(E)$ is sup-generated by $\mathcal{S} = \{\{v\} \mid v \in E\}$ (one can show that $\mathcal{S} \subseteq \mathcal{L}$, due to the Hausdorff condition). It is atomic and semi-atomic, but not strongly semi-atomic in general. Moreover, it is infinite $\wedge$-distributive, but not infinite $\vee$-distributive in general.

(c) (*Lattice of open sets*). Let E be a topological space. The collection $\mathcal{G}(E)$ of all open subsets of E is a lattice. The partial order is set inclusion, the supremum is set union, whereas the infimum is the topological interior of set intersection. Now, assume that E is a metric space. Then, $\mathcal{G}(E)$ is sup-generated by $\mathcal{S} = \{B(v, r) \mid v \in E, r > 0\}$, the nonempty open balls $B(v, r)$ of radius r, centered at v. Lattice $\mathcal{G}(E)$ is neither atomic nor strongly semi-atomic in general. Moreover, it is infinite $\vee$-distributive, but not infinite $\wedge$-distributive in general.

(d) (*Function lattice*). The collection $\mathrm{Fun}(E, \mathcal{T})$ of all functions from a set E into a lattice $\mathcal{T}$ is a lattice. The partial order is the product ordering

$$f \leq g \ \text{ if } \ f(v) \leq_{\mathcal{T}} g(v), \ \text{for all } v \in E, \tag{2.5}$$

where "$\leq_{\mathcal{T}}$" is the partial order relation on $\mathcal{T}$. The supremum and infimum are the "pointwise" infimum and supremum, given respectively by

$$(\bigvee f_\alpha)(v) = \bigvee f_\alpha(v) \tag{2.6}$$

$$(\bigwedge f_\alpha)(v) = \bigwedge f_\alpha(v), \tag{2.7}$$

for all $v \in E$, where the supremum and infimum on the right-hand side are of course in $\mathcal{T}$. Let $\mathcal{S}_{\mathcal{T}}$ be a sup-generating family for lattice $\mathcal{T}$. Lattice $\mathrm{Fun}(E, \mathcal{T})$

is sup-generated by $\mathcal{S} = \{\delta_{v,t} \mid v \in E,\ t \in \mathcal{S}_\mathcal{T}\}$, where

$$\delta_{v,t}(w) = \begin{cases} t, & \text{if } w = v \\ 0, & \text{otherwise} \end{cases}, \quad w \in E, \tag{2.8}$$

is known as a *pulse function* in $\text{Fun}(E,\mathcal{T})$. Here, "0" denotes the least element of $\mathcal{T}$. Lattice $\text{Fun}(E,\mathcal{T})$ inherits many of its properties from lattice $\mathcal{T}$. In particular, $\text{Fun}(E,\mathcal{T})$ is atomic, (strongly) semi-atomic, infinite $\vee$- or $\wedge$- distributive, for any given E, if and only if $\mathcal{T}$ enjoys the same properties.

(e) (*Lattice of upper semi-continuous functions*). Let E be a topological space. A function f from E into a lattice $\mathcal{T}$ is said to be *upper semi-continuous* (u.s.c.) if, given $v \in E$ and $t \in \mathcal{T}$ such that $t \not\leq f(v)$, then one can find a neighborhood U of v such that $t \not\leq f(w)$, for every $w \in U$. The collection $\text{Fun}_u(E,\mathcal{T})$ of all u.s.c. functions from E into $\mathcal{T}$ is a lattice. The partial order is the product ordering in (2.5), the infimum is just the pointwise infimum, whereas the supremum is the smallest (in the pointwise sense) u.s.c. function greater than the pointwise supremum. Now, assume that E is a Hausdorff space. Then $\text{Fun}_u(E,\mathcal{T})$ is sup-generated by $\mathcal{S} = \{\delta_{v,t} \mid v \in E,\ t \in \mathcal{S}_\mathcal{T}\}$, the pulses in $\text{Fun}(E,\mathcal{T})$ (one can show that $\mathcal{S} \subseteq \mathcal{L}$, due to the Hausdorff condition). It is atomic, semi-atomic and infinite $\wedge$-distributive, for any given E, if and only if $\mathcal{T}$ enjoys the same properties. However, it is neither strongly semi-atomic nor infinite $\vee$-distributive in general.

(f) (*Lattice of lower semi-continuous functions*). Let E be a topological space. A function f from E into a lattice $\mathcal{T}$ is said to be *lower semi-continuous* (l.s.c.) if, given $v \in E$ and $t \in \mathcal{T}$ such that $t \not\geq f(v)$, then one can find a neighborhood U of v such that $t \not\geq f(w)$, for every $w \in U$. The collection $\text{Fun}_l(E,\mathcal{T})$ of all l.s.c. functions from E into $\mathcal{T}$ is a lattice. The partial order is the product ordering in (2.5), the supremum is just the pointwise supremum, while the infimum is the largest (in the pointwise sense) l.s.c. function smaller than the pointwise infimum. Now, assume that E is a metric space. Then $\text{Fun}_l(E,\mathcal{T})$ is sup-generated by $\mathcal{S} = \{h_{B(v,r),t} \mid v \in E,\ r > 0,\ t \in \mathcal{T} \smallsetminus \{0\}\}$, where

$$h_{B(v,r),t}(w) = \begin{cases} t, & \text{if } w \in B(v,r) \\ 0, & \text{otherwise} \end{cases}, \quad w \in E, \tag{2.9}$$

is a *cylinder* with base the nonempty open ball $B(v,r)$ and height equal to t. Lattice

$\mathrm{Fun}_l(E,\mathcal{T})$ is neither atomic nor strongly semi-atomic in general. It is infinite $\vee$-distributive, for any given E, if and only if $\mathcal{T}$ is, but it is not infinite $\wedge$-distributive in general. $\Diamond$

The previous lattices are of special interest in image analysis. The lattices in Examples 2.1.2(a)–(c) are used as mathematical models for binary images, whereas the lattices in Examples 2.1.2(d)–(f) are used as mathematical models for grayscale images, if $\mathcal{T}$ is a chain, and multispectral images (e.g., color images), if $\mathcal{T}$ is a finite product of chains (for a suitable choice of E, this includes n-dimensional images as well as image sequences).

Since any chain $\mathcal{T}$ is semi-atomic and infinite distributive [34], the lattice $\mathrm{Fun}(E,\mathcal{T})$ of grayscale images always enjoys these properties. Similarly, if $\mathcal{T}$ is a chain, $\mathrm{Fun}_u(E,\mathcal{T})$ is semi-atomic and infinite $\wedge$-distributive, whereas $\mathrm{Fun}_l(E,\mathcal{T})$ is infinite $\vee$-distributive. For continuous-valued images, one usually sets $\mathcal{T} = \overline{\mathbb{R}}$, in which case $\mathcal{S}_\mathcal{T} = \mathbb{R}$, whereas for discrete-valued images, one usually sets $\mathcal{T} = \overline{\mathbb{Z}}$ or $\mathcal{T} = \{0,1,\ldots,R-1\}$, where $R \geq 2$ is a finite integer, in which case $\mathcal{S}_\mathcal{T} = \mathbb{Z}$ or $\mathcal{S}_\mathcal{T} = \{1,2,\ldots,R-1\}$, respectively. Both lattices $\mathrm{Fun}(E,\overline{\mathbb{Z}})$ and $\mathrm{Fun}(E,\{0,1,\ldots,R-1\})$ are strongly semi-atomic; they are not atomic, for $R \geq 3$. However, $\mathrm{Fun}(E,\overline{\mathbb{R}})$ is semi-atomic, but not strongly semi-atomic.

The lattice of grayscale images $\mathrm{Fun}(E,\mathcal{T})$ is atomic, for any given E, if and only if $\mathcal{T}$ is an atomic chain, which is true if and only if $\mathcal{T}$ is bi-valued (e.g., when $\mathcal{T} = \{0,1\}$). However, it is easy to see that $\mathrm{Fun}(E,\{0,1\})$ is isomorphic to the set lattice $\mathcal{P}(E)$. Similarly, it can be shown (as a direct consequence of Proposition 2.1.3 below) that $\mathrm{Fun}_u(E,\{0,1\})$ is isomorphic to $\mathcal{F}(E)$ and $\mathrm{Fun}_l(E,\{0,1\})$ is isomorphic to $\mathcal{G}(E)$. The lattices $\mathrm{Fun}_u(E,\mathcal{T})$ and $\mathrm{Fun}_l(E,\mathcal{T})$ extend lattices $\mathcal{F}(E)$ and $\mathcal{G}(E)$, respectively, in the same way that $\mathrm{Fun}(E,\mathcal{T})$ extends $\mathcal{P}(E)$. As a matter of fact, we have the following characterization of $\mathrm{Fun}_u(E,\mathcal{T})$ and $\mathrm{Fun}_l(E,\mathcal{T})$ (for a proof, see [34, pp. 347–348]).

2.1.3 Proposition. Let $f \in \mathrm{Fun}(E,\mathcal{T})$.

(a) $f \in \mathrm{Fun}_u(E,\mathcal{T})$ if and only if the sets $X_t(f) = \{v \in E \mid f(v) \geq t\}$ are closed in E, for all $t \in \mathcal{T}$.

(b) $f \in \mathrm{Fun}_l(E,\mathcal{T})$ if and only if the sets $Y_t(f) = \{v \in E \mid f(v) \not\leq t\}$ are open in E, for all $t \in \mathcal{T}$. $\Box$

We conclude this section by introducing the notions of underlattice and sublattice. A nonempty subset $\mathcal{M}$ of a lattice $\mathcal{L}$ is clearly a poset $(\mathcal{M},\leq)$, under the partial order

$\leq$ of $\mathcal{L}$. We say that $\mathcal{M}$ is an *underlattice* of $\mathcal{L}$ if $(\mathcal{M}, \leq)$ is a complete lattice. The supremum and infimum in $\mathcal{M}$ need not be equal to the supremum and infimum in $\mathcal{L}$. If they are equal, $\mathcal{M}$ is said to be a *sublattice* of $\mathcal{L}$. For instance, if $E' \subseteq E$, then $\mathcal{P}(E')$ is a sublattice of $\mathcal{P}(E)$. On the other hand, lattices $\mathcal{F}(E)$ and $\mathcal{G}(E)$ are underlattices of $\mathcal{P}(E)$, whereas lattices $\mathrm{Fun}_u(E, \mathcal{T})$ and $\mathrm{Fun}_l(E, \mathcal{T})$ are underlattices of $\mathrm{Fun}(E, \mathcal{T})$.

2.2 Basic Morphological Operators

Given two lattices $\mathcal{L}$ and $\mathcal{M}$, a (lattice) *operator* is a mapping $\psi\colon \mathcal{L} \to \mathcal{M}$. An operator ψ is said to be *increasing* if $A \leq B \Rightarrow \psi(A) \leq \psi(B)$. A *dilation* is an operator δ such that $\delta(\bigvee A_\alpha) = \bigvee \delta(A_\alpha)$. Similarly, an *erosion* is an operator ϵ such that $\epsilon(\bigwedge A_\alpha) = \bigwedge \epsilon(A_\alpha)$. From these definitions, it can be easily verified that dilations and erosions are increasing operators. An *adjunction* (ϵ, δ) is a pair of operators $\epsilon\colon \mathcal{L} \to \mathcal{M}$ and $\delta\colon \mathcal{M} \to \mathcal{L}$ such that

$$\delta(B) \leq A \ \Leftrightarrow\ B \leq \epsilon(A), \tag{2.10}$$

for all $A \in \mathcal{L}$, $B \in \mathcal{M}$. It can be shown that, if (ϵ, δ) is an adjunction, then ϵ is an erosion and δ is a dilation [34, Thm. 3.13]. Moreover, to any erosion ϵ corresponds a unique dilation δ such that (ϵ, δ) is an adjunction, and vice-versa.

From now on, let us consider the case in which $\mathcal{M} = \mathcal{L}$. In this case, ψ is said to be an operator on $\mathcal{L}$. The *range* $\psi(\mathcal{L}) \subseteq \mathcal{L}$ of an operator ψ on $\mathcal{L}$ is the family $\psi(\mathcal{L}) = \{\psi(A) \mid A \in \mathcal{L}\}$. The *identity* operator $\mathbf{id}$ is the operator on $\mathcal{L}$ given by $\mathbf{id}(A) = A$, for $A \in \mathcal{L}$. The *composition* $\psi_1\psi_2$ of two operators ψ_1 and ψ_2 is given by $\psi_1\psi_2(A) = \psi_1(\psi_2(A))$. We write $\psi_1 \leq \psi_2$ if $\psi_1(A) \leq \psi_2(A)$, for all $A \in \mathcal{L}$. The *supremum* $\bigvee \psi_\alpha$ of operators ψ_α is given by $(\bigvee \psi_\alpha)(A) = \bigvee \psi_\alpha(A)$. Similarly, the *infimum* $\bigwedge \psi_\alpha$ of operators ψ_α is given by $(\bigwedge \psi_\alpha)(A) = \bigwedge \psi_\alpha(A)$. An operator ν on a lattice $\mathcal{L}$ is called a *negation* if ν is a bijection that reverses ordering (i.e., $A \leq B \Leftrightarrow \nu(A) \geq \nu(B)$), such that $\nu^2 = \mathbf{id}$. When no confusion is possible, we write A^* instead of $\nu(A)$. An important example of negation is given by the *complementation* operator $\mathtt{C}$ on $\mathcal{L} = \mathcal{P}(E)$, given by $\mathtt{C}(A) = A^c = E \setminus A$, for $A \in \mathcal{P}(E)$ (note that, in this case, we write A^c instead of A^*). Given a negation on $\mathcal{L}$ and an operator ψ on $\mathcal{L}$, the *dual operator* ψ^* on $\mathcal{L}$ is given by $\psi^*(A) = (\psi(A^*))^*$, for $A \in \mathcal{L}$. Note that $\psi^{**} = \psi$; hence, the operators ψ and ψ^* are dual to each other.

An operator ψ is said to be *idempotent* if $\psi\psi(A) = \psi(A)$, for $A \in \mathcal{L}$. The *invariance domain* of ψ is defined as

$$\mathrm{Inv}(\psi) = \{A \in \mathcal{L} \mid \psi(A) = A\}. \tag{2.11}$$

Clearly, for any operator ψ, we have that $\mathrm{Inv}(\psi) \subseteq \psi(\mathcal{L})$. Moreover, ψ is idempotent if and only if $\mathrm{Inv}(\psi) = \psi(\mathcal{L})$.

An increasing and idempotent operator is said to be a (morphological) *filter*. The operator ψ is said to be *anti-extensive* if $\psi(A) \leq A$, whereas it is said to be *extensive* if $\psi(A) \geq A$, for $A \in \mathcal{L}$. An anti-extensive filter is said to be an *opening*, whereas an extensive filter is said to be a *closing*. It is easy to show that the supremum of openings is an opening, whereas the infimum of closings is a closing. In addition, it can be shown that, if (ϵ, δ) is an adjunction on $\mathcal{L}$, then $\theta = \delta\epsilon$ is an opening on $\mathcal{L}$, whereas $\phi = \epsilon\delta$ is a closing on $\mathcal{L}$ [34, Thm. 3.25]. These openings and closings are referred to as *adjunctional openings* and *adjunctional closings*, respectively.

The following result will be quite useful.

2.2.1 Proposition. Let $\mathcal{L}$ be a lattice.

(a) If γ is an opening on $\mathcal{L}$ and ψ is an increasing and anti-extensive operator on $\mathcal{L}$, then

$$\gamma \leq \psi \;\Leftrightarrow\; \mathrm{Inv}(\gamma) \subseteq \mathrm{Inv}(\psi). \tag{2.12}$$

In particular, if ψ is an opening, then $\gamma = \psi \;\Leftrightarrow\; \mathrm{Inv}(\gamma) = \mathrm{Inv}(\psi)$.

(b) If ϕ is a closing on $\mathcal{L}$ and ψ is an increasing and extensive operator on $\mathcal{L}$, then

$$\psi \leq \phi \;\Leftrightarrow\; \mathrm{Inv}(\phi) \subseteq \mathrm{Inv}(\psi). \tag{2.13}$$

In particular, if ψ is a closing, then $\phi = \psi \;\Leftrightarrow\; \mathrm{Inv}(\phi) = \mathrm{Inv}(\psi)$. □

PROOF. (a): Assume that $\gamma \leq \psi$. For $A \in \mathrm{Inv}(\gamma)$, we have $A = \gamma(A) \leq \psi(A)$, and $\psi(A) \leq A$, since ψ is anti-extensive, so that $\psi(A) = A \Rightarrow A \in \mathrm{Inv}(\psi)$. Hence, $\mathrm{Inv}(\gamma) \subseteq \mathrm{Inv}(\psi)$. Now, assume that $\mathrm{Inv}(\gamma) \subseteq \mathrm{Inv}(\psi)$. For $A \in \mathcal{L}$, we have $\gamma(A) \leq A \Rightarrow \psi\gamma(A) \leq \psi(A)$, since γ is anti-extensive and ψ is increasing. But $\gamma(A) \in \mathrm{Inv}(\gamma)$, since γ is idempotent, which implies that $\gamma(A) \in \mathrm{Inv}(\psi) \Rightarrow \gamma(A) = \psi\gamma(A) \leq \psi(A)$. Hence, $\gamma \leq \psi$. In the particular case in which ψ is an opening, then the roles of γ and ψ can be reversed in (2.12), so that $\gamma = \psi \;\Leftrightarrow\; \mathrm{Inv}(\gamma) = \mathrm{Inv}(\psi)$.

(b): This statement is the dual to the statement in part (a), and its proof is completely analogous. Q.E.D.

As a consequence of the above result, openings and closings are uniquely characterized by their domain of invariance. The reader is referred to [65, 77], for a more detailed characterization of the relationship between openings/closings and their domains of invariance.

The following result is a consequence of the so-called strong version of Tarski's fixpoint theorem [34, Prop. 12.27].

2.2.2 Proposition. Let $\mathcal{L}$ be a lattice, with infimum $\bigwedge$ and supremum $\bigvee$.

(a) If ϕ is a closing on $\mathcal{L}$, then $\mathrm{Inv}(\phi)$ is an underlattice of $\mathcal{L}$, with infimum $\bigwedge A_\alpha$ and supremum $\phi(\bigvee A_\alpha)$.

(b) If θ is an opening on $\mathcal{L}$, then $\mathrm{Inv}(\theta)$ is an underlattice of $\mathcal{L}$, with supremum $\bigvee A_\alpha$ and infimum $\theta(\bigwedge A_\alpha)$. □

A filter ψ is said to be an *inf-filter* if $\psi(\mathbf{id} \wedge \psi) = \psi$, whereas it is said to be a *sup-filter* if $\psi(\mathbf{id} \vee \psi) = \psi$. A filter that is both an inf-filter and a sup-filter is said to be a *strong filter*. It can be easily seen that ψ is a strong filter if and only if ψ satisfies the following "robustness" property:

$$A \wedge \psi(A) \leq B \leq A \vee \psi(A) \;\Rightarrow\; \psi(A) = \psi(B), \tag{2.14}$$

for all $A, B \in \mathcal{L}$. It can be shown [34, Prop. 12.3] that openings and closings are strong filters. Moreover, we have the following result (this is a direct consequence of [34, Prop. 12.5]).

2.2.3 Proposition. If θ is an opening and ϕ is a closing, then $\theta\phi$ is a sup-filter and $\phi\theta$ is an inf-filter. □

Given a family $\mathcal{M} \subseteq \mathcal{L}$, we denote by $\langle\, \mathcal{M} \mid \vee \,\rangle$ the family sup-generated by $\mathcal{M}$, i.e., the family consisting of all elements of $\mathcal{L}$ that are obtained by taking suprema of elements of $\mathcal{M}$. The family $\mathcal{M}$ is said to be *sup-closed* if $\mathcal{M} = \langle\, \mathcal{M} \mid \vee \,\rangle$ (in particular, $\mathcal{M}$ must be non-empty, since $O = \bigvee \emptyset \in \mathcal{M}$). It is easy to see that $\langle\, \mathcal{M} \mid \vee \,\rangle$ is the smallest sup-closed family that contains $\mathcal{M}$. We have the following result.

2.2.4 Proposition. If ψ is increasing and anti-extensive, then $\mathrm{Inv}(\psi)$ is sup-closed. □

PROOF. From the anti-extensivity of ψ, we have that $\psi(O) = O \Rightarrow O \in \mathrm{Inv}(\psi)$ and, therefore, $\mathrm{Inv}(\psi)$ is non-empty. Thus, consider a family $\{A_\alpha\}$ in $\mathrm{Inv}(\psi)$. From the anti-extensivity of ψ, we have that $\psi(\bigvee A_\alpha) \leq \bigvee A_\alpha$. On the other hand, since ψ is increasing, we have that $\psi(\bigvee A_\alpha) \geq \psi(A_\alpha) = A_\alpha$, for all α, so that $\psi(\bigvee A_\alpha) \geq \bigvee A_\alpha$. Hence, $\psi(\bigvee A_\alpha) = \bigvee A_\alpha \Rightarrow \bigvee A_\alpha \in \mathrm{Inv}(\psi)$ and, therefore, $\mathrm{Inv}(\psi)$ is sup-closed. Q.E.D.

To each operator ψ on a lattice $\mathcal{L}$, there corresponds an operator ψ° on $\mathcal{L}$, called the *characteristic opening* associated with ψ, given by

$$\psi^\circ(A) = \bigvee\{B \in \mathrm{Inv}(\psi) \mid B \leq A\}, \quad A \in \mathcal{L}. \tag{2.15}$$

We have the following characterization of ψ°.

2.2.5 Proposition. Let ψ be an operator on a lattice $\mathcal{L}$. The operator ψ°, defined in (2.15), is an opening on $\mathcal{L}$, with $\mathrm{Inv}(\psi^\circ) = \langle\, \mathrm{Inv}(\psi) \mid \vee\,\rangle$. □

PROOF. From (2.15), it is clear that ψ° is increasing and anti-extensive. This implies that $\psi^\circ\psi^\circ \leq \psi^\circ$. On the other hand, (2.15) implies that $B \in \mathrm{Inv}(\psi), B \leq A \Rightarrow B \leq \psi^\circ(A)$. Hence, $\psi^\circ(A) = \bigvee\{B \in \mathrm{Inv}(\psi) \mid B \leq A\} \leq \bigvee\{B \in \mathrm{Inv}(\psi) \mid B \leq \psi^\circ(A)\} = \psi^\circ\psi^\circ(A)$. Therefore, $\psi^\circ\psi^\circ = \psi^\circ$ (i.e., ψ° is idempotent), so that ψ° is an opening.

If $A \in \mathrm{Inv}(\psi^\circ)$ (i.e., if $A = \psi^\circ(A) = \bigvee\{B \in \mathrm{Inv}(\psi) \mid B \leq A\}$), then $A \in \langle\, \mathrm{Inv}(\psi) \mid \vee\,\rangle$, so that $\mathrm{Inv}(\psi^\circ) \subseteq \langle\, \mathrm{Inv}(\psi) \mid \vee\,\rangle$. On the other hand, it is clear that $\mathrm{Inv}(\psi) \subseteq \mathrm{Inv}(\psi^\circ)$. In addition, according to Proposition 2.2.4, $\mathrm{Inv}(\psi^\circ)$ is sup-closed, so that $\langle\, \mathrm{Inv}(\psi) \mid \vee\,\rangle \subseteq \mathrm{Inv}(\psi^\circ)$. This implies that $\mathrm{Inv}(\psi^\circ) = \langle\, \mathrm{Inv}(\psi) \mid \vee\,\rangle$, as required. Q.E.D.

The following result examines the relationship between an operator and its characteristic opening.

2.2.6 Proposition. For any operator ψ on $\mathcal{L}$, we have that:

(a) $\psi^\circ \leq \psi$, if ψ is increasing and anti-extensive.

(b) $\psi \leq \psi^\circ$, if ψ is anti-extensive and idempotent.

(c) $\psi = \psi^\circ$ if and only if ψ is an opening. □

PROOF. (a): By combining Propositions 2.2.4 and 2.2.5, we get $\mathrm{Inv}(\psi^\circ) = \mathrm{Inv}(\psi)$. Hence, given an $A \in \mathcal{L}$, we have that $\psi^\circ(A) \in \mathrm{Inv}(\psi^\circ) = \mathrm{Inv}(\psi) \Rightarrow \psi^\circ(A) = \psi\psi^\circ(A) \leq \psi(A)$, since ψ is increasing.

(b): Given an $A \in \mathcal{L}$, we have that $\psi(A) \in \mathrm{Inv}(\psi)$ and $\psi(A) \leq A$, from the idempotence and anti-extensivity of ψ. Hence, from (2.15), we have that $\psi(A) \leq \psi^\circ(A)$.

(c): This is a direct consequence of parts (a) and (b). Q.E.D.

The following result is a direct corollary of Propositions 2.2.1–2.2.5.

2.2.7 Corollary. If ψ is an increasing and anti-extensive operator, then $\mathrm{Inv}(\psi^\circ) = \mathrm{Inv}(\psi)$ and ψ° is the greatest opening that is smaller than ψ. □

Proof. Propositions 2.2.4 and 2.2.5 imply that $\mathrm{Inv}(\psi^\circ) = \mathrm{Inv}(\psi)$. From Proposition 2.2.1, it follows that $\psi^\circ \leq \psi$. Moreover, if γ is any opening such that $\gamma \leq \psi$ then, again from Proposition 2.2.1, we have that $\mathrm{Inv}(\gamma) \subseteq \mathrm{Inv}(\psi) = \mathrm{Inv}(\psi^\circ) \Rightarrow \gamma \leq \psi^\circ$, so that ψ° is the greatest opening smaller than ψ. Q.E.D.

Next, we define the notion of a granulometry, which is of great importance in mathematical morphology [55, 76].

2.2.8 Definition. Let $\mathcal{L}$ be a lattice, and let J be a poset. A family $\{\theta_\alpha \mid \alpha \in J\}$ of openings on $\mathcal{L}$ is said to be a *granulometry* on $\mathcal{L}$ if it is decreasing:

$$\theta_\alpha \leq \theta_\beta, \text{ for } \alpha \geq \beta. \tag{2.16}$$

The granulometry $\{\theta_\alpha \mid \alpha \in J\}$ is said to be *parameterized* by J. △

By using the fact that the members of a granulometry are openings, one can arrive easily at the following alternative characterization.

2.2.9 Proposition. Let $\mathcal{L}$ be a lattice and J be a poset. A family $\{\theta_\alpha \mid \alpha \in J\}$ of openings on $\mathcal{L}$ is a granulometry on $\mathcal{L}$ if and only if it satisfies the *semigroup property*:

$$\theta_\alpha\theta_\beta = \theta_\beta\theta_\alpha = \theta_\alpha, \text{ for } \alpha \geq \beta. \tag{2.17}$$

□

Condition (2.17) is classically described in mathematical morphology by saying that, in a granulometry, the *stronger* opening (i.e., the one with the larger parameter) "commands" the *weaker* one (i.e., the one with the smaller parameter) [76]. In other words, applying two (comparable) openings of the granulometry in either order amounts to applying only the stronger one.

An increasing operator ψ on $\mathcal{L}$ is said to be *lattice upper semi-continuous* (l.u.s.c) if, for every totally ordered subset $\mathcal{K}$ of $\mathcal{L}$, we have that

$$\psi(\bigwedge \mathcal{K}) = \bigwedge_{A \in \mathcal{K}} \psi(A). \tag{2.18}$$

If (2.18) holds when $\mathcal{K}$ contains at most a countable number of elements, then ψ is said to be $\downarrow$*-continuous*. Clearly, lattice upper semi-continuity implies $\downarrow$-continuity, but not vice-versa. Note that both definitions imply that $\psi(I) = I$ (this corresponds to the case in which $\mathcal{K} = \emptyset$).

Even though $\downarrow$-continuity is a weaker concept than lattice upper semi-continuity, it is quite useful. In particular, we have the following result.

2.2.10 Proposition. Let ψ be an $\downarrow$-continuous operator on a lattice $\mathcal{L}$, and let $\{A(s)\}_{s \in \overline{\mathbb{R}}}$ be a decreasing family of elements in $\mathcal{L}$. Then,

$$\psi(\bigwedge_{s<t} A(s)) = \bigwedge_{s<t} \psi(A(s)), \quad \text{for all } t \in \overline{\mathbb{R}}. \tag{2.19}$$

□

Proof. If $t = -\infty$, the result follows from the fact that $\psi(I) = I$. So, let $t > -\infty$. We can pick an increasing sequence $(s_i)_{i \in \mathbb{Z}_+}$ of numbers in $\overline{\mathbb{R}}$ such that $s_i \to t$. We claim that $\bigwedge_{s<t} A(s) = \bigwedge A(s_i)$. The inequality $\bigwedge_{s<t} A(s) \leq \bigwedge A(s_i)$ is obvious. To show the reverse inequality, pick an $s < t$. Since $s_i \to t$, we can find some index i_0 such that $s \leq s_i < t$, for every $i \geq i_0$, which implies that $A(s_i) \leq A(s)$, for every $i \geq i_0$. This in turn implies that $\bigwedge A(s_i) \leq A(s)$. Since this holds for any $s < t$, we have that $\bigwedge A(s_i) \leq \bigwedge_{s<t} A(s)$ and, therefore, $\bigwedge A(s_i) = \bigwedge_{s<t} A(s)$. By using a similar argument and the fact that ψ is an increasing operator, we can show that $\bigwedge_{s<t} \psi(A(s)) = \bigwedge \psi(A(s_i))$. Now, $\{A(s_i)\}$ is a countable and totally ordered family of sets, and since ψ is an $\downarrow$-continuous operator, we have that $\psi(\bigwedge_{s<t} A(s)) = \psi(\bigwedge A(s_i)) = \bigwedge \psi(A(s_i)) = \bigwedge_{s<t} \psi(A(s))$, which shows (2.19). Q.E.D.

The dual concepts of *lattice lower semi-continuous* (l.l.s.c.) and $\uparrow$*-continuous* operators are defined analogously, and similar remarks apply in this case by replacing $\bigwedge$ with $\bigvee$. If ψ is both l.u.s.c. and l.l.s.c., then ψ is said to be *lattice continuous*, whereas if ψ is both $\downarrow$-continuous and $\uparrow$-continuous, it is said to be $\updownarrow$*-continuous*.

Let us now consider the case in which $\mathcal{L} = \mathcal{P}(E)$, with $E = \mathbb{R}^n$ or $\mathbb{Z}^n$. The translation A_h of a set $A \in \mathcal{P}(E)$ is another set in $\mathcal{P}(E)$, given by $A_h = \{v + h \mid v \in A\}$. The *translation-invariant erosion* of $A \in \mathcal{P}(E)$ by a *structuring element* $B \in \mathcal{P}(E)$ is defined as

$$\epsilon_B(A) = A \ominus B = \{h \in E \mid B_h \subseteq A\}. \tag{2.20}$$

Similarly, the *translation-invariant dilation* of A by B is defined as

$$\delta_B(A) = A \oplus B = \bigcup \{B_h \mid h \in A\}. \tag{2.21}$$

It can be shown that (ϵ_B, δ_B) is an adjunction on $\mathcal{P}(E)$ [34, Prop. 4.13]. Furthermore, all translation-invariant erosions and dilations on $\mathcal{P}(E)$ are of this form. It can also be seen that ϵ_B and δ_B are dual to each other: $\epsilon_B = \delta_B^*$, or $\epsilon_B(A) = (\delta_B(A^c))^c$, for $A \in \mathcal{P}(E)$.

At times it is convenient, both from a theoretical and a practical point of view, to define the counterparts of the dilation δ_B and erosion ϵ_B on $\mathcal{P}(E)$, when E is a proper subset of $\mathbb{R}^n$ or $\mathbb{Z}^n$. Given $A, B \in \mathcal{P}(E)$, we define

$$\epsilon_B^E(A) = \epsilon_B(A) \cap E = (A \ominus B) \cap E = A \ominus_E B, \tag{2.22}$$

$$\delta_B^E(A) = \delta_B(A) \cap E = (A \oplus B) \cap E = A \oplus_E B. \tag{2.23}$$

It is easy to see that ϵ_B^E is an erosion on $\mathcal{P}(E)$ and, by infinite $\vee$-distributivity of $\mathcal{P}(E)$, δ_B^E is a dilation on $\mathcal{P}(E)$. However, the pair $(\epsilon_B^E, \delta_B^E)$ is *not* in general an adjunction on $\mathcal{P}(E)$.

The adjunctional openings and closings on $\mathcal{P}(E)$ that correspond to ϵ_B and δ_B are given by

$$\theta_B(A) = A \bigcirc B = (A \ominus B) \oplus B = \bigcup_{h \in E} \{B_h \mid B_h \subseteq A\}, \tag{2.24}$$

$$\phi_B(A) = A \bullet B = (A \oplus B) \ominus B = \left[\bigcup_{h \in E} \{B_h \mid B_h \subseteq A^c\} \right]^c. \tag{2.25}$$

The operators θ_B and ϕ_B, which are clearly dual to each other, are referred to as *structural openings* and *structural closings*, respectively. If $A \in \mathrm{Inv}(\theta_B)$, we say that A is *B-open*. Similarly, if $A \in \mathrm{Inv}(\phi_B)$, we say that A is *B-closed*. We remark that the operator $\theta_B(A) = A \bigcirc B = \bigcup_{h \in E} \{B_h \mid B_h \subseteq A\}$ defines an opening on $\mathcal{P}(E)$, even when E is a proper subset of $\mathbb{R}^n$ or $\mathbb{Z}^n$.

The operators discussed so far are binary. A useful way to build a grayscale operator is to start from a binary operator and use a technique known as *flat extension*. In the following, we review some facts related to flat extensions.

Let $\{A(t)\}_{t\in\mathcal{T}}$ be a decreasing family of sets in $\mathcal{P}(E)$. We say that function $f \in \mathrm{Fun}(E,\mathcal{T})$ is *generated* by $\{A(t)\}_{t\in\mathcal{T}}$ if

$$f(x) = \bigvee\{t \in \mathcal{T} \mid x \in A(t)\}, \quad x \in E. \tag{2.26}$$

In addition, we define the *threshold operator* X_t: $\mathrm{Fun}(E,\mathcal{T}) \to \mathcal{P}(E)$, for $t \in \mathcal{T}$, as

$$X_t(f) = \{v \in E \mid f(v) \geq t\}. \tag{2.27}$$

We say that $X_t(f)$ is the *threshold set* of f at level t. It is easy to see that f is generated by the family $\{X_t(f)\}_{t\in\mathcal{T}}$ of threshold sets.

The following facts will be useful in the sequel. We summarize them here for easy reference.

2.2.11 Proposition. Let $f \in \mathrm{Fun}(E,\mathcal{T})$ be generated by a decreasing family $\{A(t)\}_{t\in\mathcal{T}}$ of sets in $\mathcal{P}(E)$.

(a) If $\mathcal{T} = \overline{\mathbb{R}}$, then

$$X_t(f) = \bigcap_{s<t} A(s), \quad \text{for all } t \in \overline{\mathbb{R}}. \tag{2.28}$$

In particular, $X_t(f) = \bigcap_{s<t} X_s(f)$, for all $t \in \overline{\mathbb{R}}$.

(b) If $\mathcal{T} = \overline{\mathbb{Z}}$, then

$$X_t(f) = \begin{cases} E, & \text{for } t = -\infty \\ A(t), & \text{for } t \in \mathbb{Z} \\ \bigcap_{s<\infty} A(s), & \text{for } t = \infty \end{cases}. \tag{2.29}$$

(c) If $\mathcal{T} = \{0, 1, \ldots, R-1\}$, then

$$X_t(f) = \begin{cases} E, & \text{for } t = 0 \\ A(t), & \text{otherwise} \end{cases}. \tag{2.30}$$

□

2.2.12 Proposition. Let $f, g \in \mathrm{Fun}(E,\mathcal{T})$, and let $\{f_\alpha\}$ be a family of functions in $\mathrm{Fun}(E,\mathcal{T})$.

(a) $f \leq g$ if and only if $X_t(f) \subseteq X_t(g)$, for all $t \in \mathcal{T}$.

(b) $\bigwedge f_\alpha$ (resp. $\bigvee f_\alpha$) is the function in $\mathrm{Fun}(E, \mathcal{T})$ generated by the sets $\{\bigcap X_t(f_\alpha)\}_{t \in \mathcal{T}}$ (resp. $\{\bigcup X_t(f_\alpha)\}_{t \in \mathcal{T}}$). Moreover,

$$X_t(\bigwedge f_\alpha) = \bigcap X_t(f_\alpha), \quad \text{for all } t \in \mathcal{T}, \tag{2.31}$$

and

$$X_t(f_1 \vee f_2) = X_t(f_1) \cup X_t(f_2), \quad \text{for all } t \in \mathcal{T}. \tag{2.32}$$

If $\mathcal{T} = \overline{\mathbb{Z}}$ or $\mathcal{T} = \{0, 1, \ldots, R-1\}$, then (2.32) holds for arbitrary suprema, except possibly at $t = \infty$. □

It is also convenient to define the alternative threshold operator Y_t: $\mathrm{Fun}(E, \mathcal{T}) \to \mathcal{P}(E)$, for $t \in \mathcal{T}$, as

$$Y_t(f) = \{v \in E \mid f(v) \not\leq t\}. \tag{2.33}$$

Although in this section we focus on the threshold operator X_t, all the results presented have a dual formulation for the operator Y_t. For example, it is easy to see that

$$Y_t(\bigvee f_\alpha) = \bigcup Y_t(f_\alpha), \quad \text{for all } t \in \mathcal{T}, \tag{2.34}$$

and

$$Y_t(f_1 \wedge f_2) = Y_t(f_1) \cap Y_t(f_2), \quad \text{for all } t \in \mathcal{T}, \tag{2.35}$$

whereas if $\mathcal{T} = \overline{\mathbb{Z}}$ or $\mathcal{T} = \{0, 1, \ldots, R-1\}$, then (2.35) holds for arbitrary infima, except possibly at $t = -\infty$.

Consider now a family $\{\psi_t \mid t \in \mathcal{T}\}$ of increasing operators on $\mathcal{P}(E)$, which is decreasing with respect to t (i.e., $s \leq t \Rightarrow \psi_t \leq \psi_s$). We define an operator $\widetilde{\psi}$ on $\mathrm{Fun}(E, \mathcal{T})$ by assigning to $\widetilde{\psi}(f)$ the function generated by the family of sets $\{\psi_t(X_t(f))\}_{t \in \mathcal{T}}$. In other words,

$$\widetilde{\psi}(f)(v) = \bigvee \{t \in \mathcal{T} \mid v \in \psi_t(X_t(f))\}, \quad v \in E. \tag{2.36}$$

The operator $\widetilde{\psi}$ is referred to as the *semi-flat operator* generated by the family $\{\psi_t \mid t \in \mathcal{T}\}$. It can be easily seen that $\widetilde{\psi}$ is an increasing operator.

In the particular case when $\psi_t = \psi$, for every $t \in \mathcal{T}$ (i.e., the family consists of a single operator ψ), (2.36) reduces to

$$\overline{\psi}(f)(v) = \bigvee \{t \in \mathcal{T} \mid v \in \psi(X_t(f))\}, \quad v \in E. \tag{2.37}$$

This is known as the *flat operator* generated by ψ.

In the following, we consider the cases when $\mathcal{T} = \overline{\mathbb{R}}$, $\mathcal{T} = \overline{\mathbb{Z}}$, and $\mathcal{T} = \{0, 1, \ldots, R-1\}$. In these cases, the operators $\widetilde{\psi}$ and $\overline{\psi}$ in (2.36) and (2.37) are referred to as the *semi-flat grayscale operator* generated by $\{\psi_t\}$ and the *flat grayscale operator* generated by ψ, respectively. When $\mathcal{T} = \overline{\mathbb{R}}$, Proposition 2.2.11(a) implies that

$$X_t(\overline{\psi}(f)) = \bigcap_{s<t} \psi(X_s(f)), \quad \text{for all } t \in \overline{\mathbb{R}}. \tag{2.38}$$

If ψ is $\downarrow$-continuous on $\mathcal{P}(E)$, then Proposition 2.2.10 implies that (2.38) reduces to

$$X_t(\overline{\psi}(f)) = \psi(X_t(f)), \quad \text{for all } t \in \overline{\mathbb{R}}. \tag{2.39}$$

In this case, the threshold sets $X_t(\overline{\psi}(f))$ of function $\overline{\psi}(f)$ can be computed by simply applying the binary operator ψ to the corresponding threshold sets $X_t(f)$ of function f. As a direct consequence of Proposition 2.2.11(b),(c), (2.39) also holds in the cases when $\mathcal{T} = \overline{\mathbb{Z}}$ and $\mathcal{T} = \{0, 1, \ldots, R-1\}$ (except possibly at $t = \infty$), provided that $\psi(E) = E$.

Semi-flat grayscale operators preserve many useful properties of the original binary operators that generate them. We have the following result, which follows from Corollary 10.26 in [34].

2.2.13 Proposition.

(a) If $\{\psi_t\}$ is a family of erosions (resp., dilations, openings, closings), then $\widetilde{\psi}$, generated by $\{\psi_t\}$, is an erosion (resp., dilation, opening, closing).

(b) If ψ is an erosion (resp., dilation, opening, closing), then $\overline{\psi}$, generated by ψ, is an erosion (resp., dilation, opening, closing). □

In particular, the translation-invariant erosion $\epsilon_B(A) = A \ominus B$ and the translation-invariant dilation $\delta_B(A) = A \oplus B$ generate the grayscale erosion $\overline{\epsilon}_B(f) = f \ominus B$, known as the *flat grayscale translation-invariant erosion*, and the grayscale dilation $\overline{\delta}_B(f) = f \oplus B$, known as the *flat grayscale translation-invariant dilation*, respectively (for simplicity, we write $\epsilon_B(f)$ and $\delta_B(f)$ instead of $\overline{\epsilon}_B(f)$ and $\overline{\delta}_B(f)$). Similarly, we can define the *flat grayscale structural opening* $\theta_B(f) = f \circ B$ and the *flat grayscale structural closing* $\phi_B(f) = f \bullet B$.

2.3 Basic Topological Concepts

Our objective in this section is to summarize basic facts about topological spaces, to be used later in this dissertation. Most of the results are given without proof. Note, however, that this is not intended to be a detailed introduction on topology. For that, the reader is referred to [24, 60].

2.3.1 Definition. Given a set E, a *topological space* is a pair $(E, \mathcal{G})$, where $\mathcal{G} \subseteq \mathcal{P}(E)$ is such that:

(i) $\emptyset$ and E are in $\mathcal{G}$,

(ii) an arbitrary union of elements in $\mathcal{G}$ is in $\mathcal{G}$,

(iii) a finite intersection of elements in $\mathcal{G}$ is in $\mathcal{G}$.

The family $\mathcal{G}$ is said to be a *topology* on E. $\triangle$

One usually refers to the topological space E, when no confusion as to the underlying topology $\mathcal{G}$ arises. Sometimes, we write $\mathcal{G}(E)$ instead of $\mathcal{G}$. The elements of $\mathcal{G}$ are said to be the *open* sets of E. A set $A \in \mathcal{P}(E)$ is said to be a *closed* set of E if A^c is open. Given a point $v \in E$, a *neighborhood* of v is an open set U that contains v. The *interior* A° of a set A is defined as the largest open set contained in A (i.e., the union of all open sets contained in A), whereas the *closure* $\overline{A}$ of A is the smallest closed set that contains A (i.e., the intersection of all closed sets that contain A).

A family $\mathcal{B} \subseteq \mathcal{G}$ is said to be a *basis* for the topology $\mathcal{G}$ on E if every open set can be written as a union of elements of $\mathcal{B}$. In this case, $\mathcal{G}$ is said to be *generated* by $\mathcal{B}$. Similarly, a family $\mathcal{U} \subseteq \mathcal{G}$ is said to be a *subbasis* for the topology $\mathcal{G}$ on E if every open set can be written as a union of finite intersections of elements of $\mathcal{U}$. In this case, $\mathcal{G}$ is also said to be *generated* by $\mathcal{U}$. Note that the family consisting of all finite intersections of elements of $\mathcal{U}$ is a basis for $\mathcal{G}$. In addition, the topology generated by a basis $\mathcal{B}$ (resp. subbasis $\mathcal{U}$) is the smallest topology that contains $\mathcal{B}$ (resp. $\mathcal{U}$).

If E is any set, then the collection $\mathcal{P}(E)$ of all subsets of E is a topology on E. This is called the *discrete topology* on E. In this case, any subset of E is both open and closed. The set of points in E forms a basis for the discrete topology. Another example is given by the topology $\mathcal{G} = \{\emptyset, E\}$. This is known as the *indiscrete topology* on E. A more useful example however is the topology $\mathcal{G}_e$ on $\mathbb{R}^n$ generated by the *open balls* $B(v, r) = \{u \in \mathbb{R}^n \mid d(u, v) < r\}$, for $v \in \mathbb{R}^n$ and $r > 0$, where d is the Euclidean distance in $\mathbb{R}^n$. This is known as the *Euclidean topology* on $\mathbb{R}^n$, and the topological space $(\mathbb{R}^n, \mathcal{G}_e)$ is known as the *n-dimensional Euclidean space.*

Two of the most important topological concepts are the notions of convergence and continuity. A *sequence* $(v_i)_{i\in\mathbb{Z}_+}$ of points in a topological space E is said to *converge* to a point $v \in E$ if, for every neighborhood U of v, there exists an index i_0 such that $v_i \in U$, for all $i \geq i_0$. In this case, we write $v_i \to v$. We say that v is an *accumulation point* of (v_i) if, for every neighborhood U of v and every index i_0, there exists an $i \geq i_0$ such that $v_i \in U$. Clearly, v is an accumulation point of (v_i) if there is a *subsequence* (v_{i_k}) of (v_i) such that $v_{i_k} \to v$, as $k \to \infty$. If E and F are topological spaces, then a function $f\colon E \to F$ is said to be *continuous* if, for any open set V in F, the set $f^{-1}(V) = \{v \in E \mid f(v) \in V\}$ is open in E.

Arbitrary topological spaces are not very useful. One needs to impose certain restrictions on them. In the following, we list some of the most frequently imposed restrictions.

2.3.2 Definition. Let $(E, \mathcal{G})$ be a topological space.

(a) $(E, \mathcal{G})$ is said to be a *Hausdorff space* if, given two distinct points $v_1, v_2 \in E$, there exist neighborhoods U_1 of v_1 and U_2 of v_2 such that $U_1 \cap U_2 \neq \emptyset$.

(b) $(E, \mathcal{G})$ is said to have a *countable basis* if the topology $\mathcal{G}$ has a basis consisting of a countable number of elements.

(c) $(E, \mathcal{G})$ is said to be a *metric space* if the topology $\mathcal{G}$ is generated by nonempty open balls $B(v, r) = \{u \in E \mid d(u, v) < r\}$, for $v \in E$ and $r > 0$, where d is a *distance function* in E [39].

(d) $(E, \mathcal{G})$ is said to be a *compact space* if every *open cover* of E (i.e., a family $\{U_\alpha\}$ of open subsets of E such that $E = \bigcup U_\alpha$), has a *finite subcover* (i.e., open sets $U_1, \ldots, U_n \in \{U_\alpha\}$ such that $E = U_1 \cup \cdots \cup U_n$). △

We now make a few remarks. In a Hausdorff space, all points are closed sets. It is obvious that every metric space is Hausdorff and has a countable basis. One of the facts that make compactness very useful is that every sequence in a compact space must have at least one accumulation point in that space. In the case of a metric space, this property is equivalent to compactness. The Euclidean space is a metric space. It is therefore Hausdorff and has a countable basis, but it is not compact, since $\{B(0, r) \mid r > 0\}$ is an open cover of $\mathbb{R}^n$ that has no finite subcover.

Topological spaces with countable bases are very useful since they allow a straightforward characterization for convergence and continuity. This is clear from the following proposition.

2.3.3 Proposition. Let E be a topological space that has a countable basis.

(a) If $A \subseteq E$, then $v \in \overline{A}$ if and only if there is a sequence (v_i) in A such that $v_i \to v$.

(b) A function $f\colon E \to F$ is continuous if and only if, for every sequence (v_i) in E such that $v_i \to v$, we have that $f(v_i) \to f(v)$. □

Given a topological space $(E, \mathcal{G})$, let A be a subset of E. Then, the family $\mathcal{G} \cap A = \{U \cap A \mid U \in \mathcal{G}\}$ is a topology on A, known as the *subspace topology*. The space $(A, \mathcal{G} \cap A)$ is called a *topological subspace* of E. Any property of a topological space, such as the ones listed in Definition 2.3.2, applies to the topological subspace via the subspace topology. A subset $A \subseteq E$ is said to be *compact* if the corresponding topological subspace $(A, \mathcal{G} \cap A)$ is a compact space.

It is easy to see that any topological subspace of a Hausdorff space with a countable basis is also Hausdorff with a countable basis. As for compactness, we have the following result.

2.3.4 Proposition.

(a) Any closed subset of a compact space is compact.

(b) Any compact subset of a Hausdorff space is closed. □

As a corollary, a subset of a compact Hausdorff space is compact if and only if it is closed. As we have seen, the Euclidean space is not compact, so a closed subset in $\mathbb{R}^n$ need not be compact. A characterization of the compact subsets of the Euclidean space is given by the following result. A subset A in $\mathbb{R}^n$ is *bounded* if there is a ball $B(0, r)$ with finite radius that contains A.

2.3.5 Proposition. A subset of the Euclidean space is compact if and only if it is closed and bounded. □

In particular, note that any closed and bounded subset of $\mathbb{R}^n$ defines a compact Hausdorff subspace with countable basis.

Compact Hausdorff spaces are nice (they belong to a class of spaces known as "normal spaces"), in the sense that closed sets behave as points. This is stated by the following proposition.

2.3.6 Proposition. Let E be a compact Hausdorff space. Given any two nonempty disjoint closed subsets A_1, A_2 of E, there are disjoint open subsets U_1, U_2 of E such that $A_1 \subset U_1$ and $A_2 \subset U_2$. □

We conclude this subsection with the following result (since this is not a basic fact of topology, a proof is supplied).

2.3.7 Proposition. Let E be a compact space and $\{A_\alpha\}$ be an arbitrary decreasing family of nonempty closed sets in E.

(a) The set $\bigcap A_\alpha$ is nonempty.

(b) If U is an open set and $\bigcap A_\alpha \subset U$, then there is some index α' such that $A_{\alpha'} \subset U$. □

PROOF. (a): Suppose that $\bigcap A_\alpha = \emptyset$. If $U_\alpha = A_\alpha^c$, then $\{U_\alpha\}$ is a family of open subsets of E such that $\bigcup U_\alpha = (\bigcap A_\alpha)^c = E$; i.e., $\{U_\alpha\}$ is an open cover of E. But, clearly, no finite subcover exists. This contradicts the assumption that E is compact. Hence, we must have that $\bigcap A_\alpha \neq \emptyset$.

(b): Let $F_\alpha = A_\alpha \setminus U$. If $A_\alpha \not\subset U$, for all indices α, then $\{F_\alpha\}$ is a decreasing family of nonempty closed subsets of E. By using part (a), we have that $\emptyset \neq \bigcap F_\alpha = \bigcap (A_\alpha \setminus U) = (\bigcap A_\alpha) \setminus U \Rightarrow \bigcap A_\alpha \not\subset U$, which is a contradiction. Hence, there must be some index α' such that $A_{\alpha'} \subset U$. Q.E.D.

2.4 Hit-or-Miss Topology

The hit-or-miss topology is a topology on the family $\mathcal{F}(E)$ of closed sets of a topological space E. By assigning a topology to $\mathcal{F}(E)$, one is able to study convergence of closed sets and (semi-) continuity properties of (closed) set-valued functions. In this section, our objective is to provide a brief introduction to the hit-or-miss topology. For a comprehensive treatment, the reader is referred to [34, 56].

Unless otherwise specified, we assume that the space E is a compact Hausdorff space with a countable basis (e.g., E may be a closed and bounded subset of $\mathbb{R}^n$ furnished with the Euclidean topology). The following is an intuitive definition of set convergence.

2.4.1 Definition. Given a sequence (A_i) of nonempty sets in $\mathcal{F}(E)$, the *limit superior* $\overline{\lim}\, A_i$ and the *limit inferior* $\underline{\lim}\, A_i$ are defined as

$$\overline{\lim}\, A_i = \{v \in E \mid \text{there exists a subsequence } (v_{i_k} \in A_{i_k}) \text{ s.t. } v_{i_k} \to v\}, \tag{2.40}$$

$$\underline{\lim}\, A_i = \{v \in E \mid \text{there exists a sequence } (v_i \in A_i) \text{ s.t. } v_i \to v\}. \tag{2.41}$$

△

Clearly, $\overline{\lim}\, A_i$ is the collection of all accumulation points of sequences $(v_i \in A_i)$, whereas $\underline{\lim}\, A_i$ is the collection of all limit points of sequences $(v_i \in A_i)$. It is obvious that $\underline{\lim}\, A_i \subseteq \overline{\lim}\, A_i$. If $\underline{\lim}\, A_i = \overline{\lim}\, A_i = A$, then the sequence (A_i) is said to *converge* to A and we write $\lim A_i = A$. For properties and examples of these limiting operations, the reader is referred to [34, Chapter 7].

It so happens that this notion of set convergence corresponds to convergence in a topology on $\mathcal{F}(E)$, called the hit-or-miss topology [34, 56].

2.4.2 Definition. Given a topological space $(E, \mathcal{G})$ with a countable basis that is compact and Hausdorff, the *hit-or-miss* (H-M) *topology* on $\mathcal{F}(E)$ is the topology generated by the subbasis $\{\mathcal{F}^F \mid F \in \mathcal{F}\} \cup \{\mathcal{F}_G \mid G \in \mathcal{G}\}$, where

$$\mathcal{F}^F = \{A \in \mathcal{F}(E) \mid A \cap F = \emptyset\}, \tag{2.42}$$

$$\mathcal{F}_G = \{A \in \mathcal{F}(E) \mid A \cap G \neq \emptyset\}. \tag{2.43}$$

△

In other words, the hit-or-miss topology is the smallest topology on $\mathcal{F}(E)$ such that each family in $\mathcal{F}(E)$ whose elements *miss* a closed set or *hit* an open set is open in $\mathcal{F}(E)$. We remark that the hit-or-miss topology has a more general formulation than the one adopted here, where compactness is relaxed to local compactness, see [34]. As formulated here, the hit-or-miss topology reduces to a topology that is known by several names in the literature, such as H-topology, exponential topology, and Vietoris topology. See [41, Section 17] and the bibliographical notes in [34, Section 7.8].

Convergence of a sequence (A_i) of closed sets to a closed set A in the H-M topology is denoted by $A_i \xrightarrow{\mathcal{F}} A$. The following result follows from Propositions 7.25 and 7.26 in [34].

2.4.3 Proposition. Given a sequence (A_i) of closed sets, we have that $A_i \xrightarrow{\mathcal{F}} A$ if and only if $\lim A_i = A$. □

It can be shown that the H-M topology has a countable basis [34, Theorem 7.24]. Therefore, by Proposition 2.3.3, sequential convergence is appropriate to characterize closure of families in $\mathcal{F}(E)$ and continuity of operators on $\mathcal{F}(E)$.

In this dissertation, we are interested in the semi-continuity properties of functions $f\colon A \to \mathcal{F}(E)$ from a subspace $A \subseteq E$ into $\mathcal{F}(E)$. Note that, since E has a countable basis, it follows from Proposition 2.3.3(b) that f is continuous if and only if $v_i \to v$ in A implies that $f(v_i) \stackrel{\mathcal{F}}{\to} f(v)$. In similar fashion to the continuity of real-valued functions, we have the following definition.

2.4.4 Definition. A function $f\colon A \to \mathcal{F}(E)$ is said to be *hit-or-miss upper semi-continuous* (H-M u.s.c) if

$$v_i \to v \;\Rightarrow\; \overline{\lim}\, f(v_i) \subseteq f(v), \tag{2.44}$$

whereas f is said to be *hit-or-miss lower semi-continuous* (H-M l.s.c.) if

$$v_i \to v \;\Rightarrow\; f(v) \subseteq \underline{\lim}\, f(v_i). \tag{2.45}$$

△

Obviously, f is continuous or H-M continuous, if and only if f is both H-M u.s.c. and H-M l.s.c. The next result follows from Propositions 7.25 and 7.26 in [34].

2.4.5 Proposition. Let $f\colon A \to \mathcal{F}(E)$ be a function from a subspace $A \subseteq E$ into $\mathcal{F}(E)$.

(a) The function f is H-M u.s.c. if and only if, for every closed subset F of E and every sequence (v_i) converging to v in A such that $f(v) \cap F = \emptyset$, we have that $f(v_i) \cap F = \emptyset$, eventually.

(b) The function f is H-M l.s.c. if and only if, for every open subset G of E and every sequence (v_i) converging to v in A such that $f(v) \cap G \neq \emptyset$, we have that $f(v_i) \cap G \neq \emptyset$, eventually. □

In other words, f is H-M u.s.c. if and only if, given a closed set F and $v_i \to v$ in A such that $f(v)$ misses F, there is only a finite number of indices for which $f(v_i)$ does not miss F. A similar remark applies to an H-M l.s.c. function.

2.5 Fuzzy Sets and Fuzzy Topological Spaces

Fuzzy sets generalize the notion of ordinary sets via the idea of a *degree of membership* to a reference set. In the original definition, due to L. Zadeh [96], a *fuzzy set* is a function μ from a reference set E into the real interval $[0,1]$. The degree of membership of a point $v \in E$ to the fuzzy set μ is given by $\mu(v)$. Hence, μ is also referred to as the *membership function*. This generalizes the notion of the *characteristic function* c_A: $E \to \{0,1\}$ of a *crisp set* $A \subseteq E$, given by

$$c_A(v) = \begin{cases} 1, & \text{if } v \in A \\ 0, & \text{if } v \in E \smallsetminus A \end{cases}. \tag{2.46}$$

Hence, crisp sets are special cases of fuzzy sets.

It has been recognized that the definition of a fuzzy set can be extended to include functions from E into any given infinite distributive lattice $\mathcal{T}$ furnished with a negation [26, 63]. This leads to the following definition.

2.5.1 Definition. Let E be a reference set and $\mathcal{T}$ an infinite distributive lattice furnished with a negation. A *$\mathcal{T}$-fuzzy subset* of E is a function μ: $E \to \mathcal{T}$. The collection of all $\mathcal{T}$-fuzzy subsets of E is denoted by $\mathcal{T}^E$. △

For instance, we may choose $\mathcal{T}$ to be $\overline{\mathbb{Z}}$, $\overline{\mathbb{R}}$, or products of those. Of course, fuzzy sets in the Zadeh sense correspond to the special case $\mathcal{T} = [0,1]$. Crisp sets in the $\mathcal{T}$-fuzzy case are fuzzy sets that take values in $\{O, I\}$, where O and I are the least and the greatest elements of $\mathcal{T}$, respectively. For convenience, and if there is no risk of confusion, we sometimes refer to $\mathcal{T}$-fuzzy sets as simply fuzzy sets. In addition, we usually denote a fuzzy set that happens to be crisp by A and a constant fuzzy set that takes on value τ on E by τ, for $\tau \in \mathcal{T}$.

A *$\mathcal{T}$-fuzzy point* v_τ is a fuzzy set that takes on value $\tau \in \mathcal{T} \smallsetminus \{O\}$ at $v \in E$, and O everywhere else. A crisp point $v \in E$ corresponds to the special case $v = v_I$.

The *fuzzy inclusion* relation between two fuzzy sets $\mu_1, \mu_2 \in \mathcal{T}^E$ is defined by the usual partial ordering of functions: $\mu_1 \le \mu_2$ if $\mu_1(v) \le \mu_2(v)$, for all $v \in E$. It is clear that $\mathcal{T}^E$ with the fuzzy inclusion relation is an infinite distributive lattice. The *fuzzy union* and *fuzzy intersection* of a family $\{\mu_\alpha\}$ of fuzzy sets in $\mathcal{T}^E$ are given by the usual pointwise supremum $\bigvee \mu_\alpha$ and pointwise infimum $\bigwedge \mu_\alpha$, respectively. The *fuzzy complement* μ^* of a fuzzy set $\mu \in \mathcal{T}^E$ is defined by $\mu^*(v) = (\mu(v))^*$, for $v \in E$, where $*$ denotes the negation in lattice $\mathcal{T}$.

Given a fuzzy set $\mu \in \mathcal{T}^E$, one can define crisp subsets of E by means of

$$X_\tau(\mu) = \{v \in E \mid \mu(v) \geq \tau\} \quad \text{and} \quad Y_\tau(\mu) = \{v \in E \mid \mu(v) \not\leq \tau\}, \tag{2.47}$$

for every $\tau \in \mathcal{T}$. The sets $X_\tau(\mu)$ and $Y_\tau(\mu)$ are level sets of the fuzzy set μ. Note that, when $\mathcal{T}$ is a chain, we have that $Y_\tau(\mu) = \{v \in E \mid \mu(v) > \tau\}$, for $\tau \in \mathcal{T}$.

The concept of fuzzy sets has led to a theory of topological fuzzy spaces. The original definition, which we adopt here, is due to C. L. Chang [18].

2.5.2 Definition. Given a set E, a $\mathcal{T}$*-fuzzy topological space* is a pair (E, Δ), where $\Delta \subseteq \mathcal{T}^E$ is such that:

(*i*) the constant fuzzy sets O and I are in Δ,

(*ii*) an arbitrary fuzzy union of elements in Δ is in Δ,

(*iii*) a finite fuzzy intersection of elements in Δ is in Δ.

The family Δ is said to be a $\mathcal{T}$*-fuzzy topology* on E. △

If there is no confusion as to the underlying lattice $\mathcal{T}$, we simply use the terms fuzzy topological space and fuzzy topology. Note that the definition of a fuzzy topological space is formally identical to the definition of an ordinary topological space (see Definition 2.3.1). The elements of Δ are said to be the open fuzzy sets of E. A fuzzy set $\mu \in \mathcal{T}^E$ is closed if μ^* is open. Note that an ordinary topological space is a fuzzy topological space where all open and closed sets are crisp (hence, fuzzy topological spaces provide a natural generalization of the notion of ordinary topological spaces). The concepts of interior, closure, basis, subbasis, indiscrete and discrete topologies are defined as in the ordinary case. Given a fuzzy topological space (E, Δ) and a crisp subset $A \subseteq E$, the family $\Delta_A = \{\mu|A \mid \mu \in \Delta\}$ is a fuzzy topology on A, where $\mu|A$ denotes the restriction of μ to A. The pair (A, Δ_A) is said to define a *fuzzy subspace* of E. The notions of continuity, convergence, Hausdorff space, compactness and other important topological properties can be defined for fuzzy topological spaces as well [18, 50, 61].

As an example, if $(E, \mathcal{G})$ is an ordinary topological space, it is easy to verify that the set of l.s.c. functions μ: $E \to \mathcal{T}$ defines a fuzzy topology on E. This fuzzy topology, denoted by $\Delta(\mathcal{G})$, is said to be *topologically generated* by $\mathcal{G}$, and the fuzzy space $(E, \Delta(\mathcal{G}))$ is said to be topologically generated [49].

Given a fuzzy topological space (E, Δ), one can define a family of sets in E by means of

$$Y_\tau(\Delta) = \{Y_\tau(\mu) \mid \mu \in \Delta\}, \tag{2.48}$$

for $\tau \in \mathcal{T}$. The following result is easy to prove.

2.5.3 Proposition. Let (E, Δ) be a $\mathcal{T}$-fuzzy topological space. For each $\tau \in \mathcal{T} \smallsetminus \{I\}$, we have that $(E, Y_\tau(\Delta))$ is an ordinary topological space. □

The topological spaces $(E, Y_\tau(\Delta))$ are called the *τ-level topological spaces* associated with the fuzzy topological space (E, Δ) [50]. The τ-level topological spaces are often used to define fuzzy topological properties; e.g., one can define τ-Hausdorff spaces, or τ-compact spaces in terms of the corresponding ordinary properties defined for the τ-level topological spaces [52].

Note that all τ-level topological spaces of an ordinary topological space $(E, \mathcal{G})$ coincide with $(E, \mathcal{G})$. It is easy to verify that all τ-level topological spaces of a topologically generated fuzzy space $(E, \Delta(\mathcal{G}))$ also coincide with $(E, \mathcal{G})$. In this sense, a topologically generated fuzzy space has the same properties of the original topological space.

As a matter of fact, in view of Proposition 2.1.3(b), it becomes clear that topologically generated fuzzy spaces are special cases of the following general result (this is an extension of [93, Proposition 3.3], which addresses the case $\mathcal{T} = [0, 1]$).

2.5.4 Proposition. Let $\mathcal{T}$ be a lattice, and let $\mathbf{G} = \{\mathcal{G}_\tau \mid \tau \in \mathcal{T} \smallsetminus \{I\}\}$ be a family of topologies on a space E. We have that

$$\Delta(\mathbf{G}) = \{\mu \in \mathcal{T}^E \mid Y_\tau(\mu) \in \mathcal{G}_\tau, \text{ for all } \tau \in \mathcal{T} \smallsetminus \{I\}\} \tag{2.49}$$

is a $\mathcal{T}$-fuzzy topology on E. Moreover, it is the largest $\mathcal{T}$-fuzzy topology Δ on E such that

$$Y_\tau(\Delta) \subseteq \mathcal{G}_\tau, \text{ for each } \tau \in \mathcal{T} \smallsetminus \{I\}. \tag{2.50}$$

□

PROOF. Clearly, the constant fuzzy sets O and I are in $\Delta(\mathbf{G})$. For a family $\{\mu_\alpha\} \subseteq \Delta(\mathbf{G})$, we have that $Y_\tau(\bigvee \mu_\alpha) = \bigcup Y_t(\mu_\alpha) \in \mathcal{G}_\tau$, since $\{Y_t(\mu_\alpha)\} \subseteq \mathcal{G}_\tau$, for each $\tau \in \mathcal{T} \smallsetminus \{I\}$, so that $\bigvee \mu_\alpha \in \Delta(\mathbf{G})$. Similarly, for $\mu_1, \mu_2 \in \Delta(\mathbf{G})$, we have that $Y_\tau(\mu_1 \wedge \mu_2) = Y_t(\mu_1) \cap Y_t(\mu_2) \in \mathcal{G}_\tau$, since $Y_t(\mu_1), Y_t(\mu_2) \in \mathcal{G}_\tau$, for each $\tau \in \mathcal{T} \smallsetminus \{I\}$, so that $\mu_1 \wedge \mu_2 \in \Delta(\mathbf{G})$. Hence, $\Delta(\mathbf{G})$ is a $\mathcal{T}$-fuzzy topology on E. Now, for a given $\tau \in \mathcal{T} \smallsetminus \{I\}$, let $A \in Y_\tau(\Delta(\mathbf{G}))$; i.e.,

$A = Y_\tau(\mu)$, for some $\mu \in \Delta(\mathbf{G})$. It follows from the definition of $\Delta(\mathbf{G})$ that $A \in \mathcal{G}_\tau$. Hence, $Y_\tau(\Delta(\mathbf{G})) \subseteq \mathcal{G}_\tau$, for each $\tau \in \mathcal{T} \setminus \{I\}$. Finally, let Δ be a $\mathcal{T}$-fuzzy topology on E such that $Y_\tau(\Delta) \subseteq \mathcal{G}_\tau$, for each $\tau \in \mathcal{T} \setminus \{I\}$. If $\mu \in \Delta$, then $Y_\tau(\mu) \in \mathcal{G}_\tau$, for each $\tau \in \mathcal{T} \setminus \{I\}$, so that $\mu \in \Delta(\mathbf{G})$. Hence, $\Delta \subseteq \Delta(\mathbf{G})$. Q.E.D.

In the case of a topologically generated fuzzy space, we have that $\mathbf{G} = \{\mathcal{G}_\tau = \mathcal{G} \mid \tau \in \mathcal{T} \setminus \{I\}\}$, where $\mathcal{G}$ is a topology on E, and $\Delta(\mathbf{G}) = \Delta(\mathcal{G})$. In this case, the inequality (2.50) becomes an equality. For a simple counterexample that shows that the inclusion in (2.50) is in general strict, let $\mathcal{T} = \overline{\mathbb{R}}$, and define the family of topologies $\mathbf{G} = \{\mathcal{G}_\tau \mid \tau \in \overline{\mathbb{R}} \setminus \{\infty\}\}$ on E by $\mathcal{G}_\tau = \{\emptyset, E\}$, if τ is rational, or $\mathcal{G}_\tau = \mathcal{G} \supset \{\emptyset, E\}$, otherwise, where $\mathcal{G}$ is any topology on E, larger than $\{\emptyset, E\}$. It is easy to show that the $\overline{\mathbb{R}}$-fuzzy topology $\Delta(\mathbf{G})$ generated by $\mathbf{G}$ contains only the constant $\overline{\mathbb{R}}$-fuzzy sets. Hence, for irrational τ, we have that $Y_\tau(\Delta(\mathbf{G})) = \{\emptyset, E\} \subset \mathcal{G} = \mathcal{G}_\tau$ (this example is adapted from [93]).

Next, we give a nontrivial example where equality in (2.50) is achieved. First, we need to define the concept of a topology pyramid.

2.5.5 Definition. A *topology pyramid* on E is a family $\mathbf{P} = \{\mathcal{G}_\tau \mid \tau \in \overline{\mathbb{R}} \setminus \{\infty\}\}$ of topologies on E such that:

(*i*) the topologies are increasingly coarser; i.e., $\mathcal{G}_{\tau_1} \subseteq \mathcal{G}_{\tau_2}$, for $\tau_2 \leq \tau_1$,

(*ii*) $\mathcal{G}_\tau = \bigcup_{s>\tau} \mathcal{G}_s$, for all $\tau \in \overline{\mathbb{R}} \setminus \{\infty\}$.

The topologies $\mathcal{G}_\tau$ are said to be the τ-*levels* of the topology pyramid $\mathbf{P}$, for $\tau \in \overline{\mathbb{R}} \setminus \{\infty\}$. △

We have the following result.

2.5.6 Proposition. Let $\mathbf{P} = \{\mathcal{G}_\tau \mid \tau \in \overline{\mathbb{R}} \setminus \{\infty\}\}$ be a topology pyramid on a space E. We have that

$$\Delta(\mathbf{P}) = \{\mu \in \overline{\mathbb{R}}^E \mid Y_\tau(\mu) \in \mathcal{G}_\tau, \text{ for all } \tau \in \overline{\mathbb{R}} \setminus \{\infty\}\} \tag{2.51}$$

is the largest $\overline{\mathbb{R}}$-fuzzy topology Δ on E such that

$$Y_\tau(\Delta) = \mathcal{G}_\tau, \text{ for each } \tau \in \overline{\mathbb{R}} \setminus \{\infty\}. \tag{2.52}$$

□

Proof. We only need to show that $\mathcal{G}_\tau \subseteq Y_\tau(\Delta(\mathbf{P}))$, for each $\tau \in \overline{\mathbb{R}} \setminus \{\infty\}$, since the rest follows from Proposition 2.5.4. Given $\tau \in \overline{\mathbb{R}} \setminus \{\infty\}$, let $G \in \mathcal{G}_\tau$. From property (*ii*) of topology pyramids, we have that $G \in \bigcup_{s>\tau} \mathcal{G}_s \Rightarrow G \in \mathcal{G}_{\tau_0}$, for some $\tau_0 > \tau$. Since $\mathbf{P}$ is decreasing, we have that $G \in \mathcal{G}_s$, for all $s < \tau_0$. Now, consider the fuzzy set $\mu \in \overline{\mathbb{R}}^E$, given by

$$\mu(v) = \begin{cases} \tau_0, & \text{if } v \in G \\ -\infty, & \text{otherwise} \end{cases}, \tag{2.53}$$

for $v \in E$. It is clear that

$$Y_s(\mu) = \begin{cases} G, & \text{if } s < \tau_0 \\ \emptyset, & \text{otherwise} \end{cases}, \tag{2.54}$$

so that $Y_s(\mu) \in \mathcal{G}_s$, for all $s \in \overline{\mathbb{R}} \setminus \{\infty\}$; i.e., $\mu \in \Delta(\mathbf{P})$. In addition, $Y_\tau(\mu) = G$, so that $G \in Y_\tau(\Delta(\mathbf{P}))$. Hence, $\mathcal{G}_\tau \subseteq Y_\tau(\Delta(\mathbf{P}))$, as required. Q.E.D.

Proposition 2.5.6 is a specialization to the case of topology pyramids of a more general result, due to P. Wuyts [93, Thm. 4.3], which gives necessary and sufficient conditions on a family of topologies $\mathbf{G} = \{\mathcal{G}_\tau \mid \tau \in [0,1)\}$ such that the $[0,1]$-fuzzy topology $\Delta(\mathbf{G})$ generated by $\mathbf{G}$ satisfies $Y_\tau(\Delta(\mathbf{G})) = \mathcal{G}_\tau$, for each $\tau \in [0,1)$ (this result can be extended from $[0,1]$ to $\overline{\mathbb{R}}$ without difficulty).

Chapter 3

Classical Connectivity

As we mentioned in Chapter 1, connectivity is classically defined using a topological or a graph-theoretic framework, and their fuzzy analogs. In this chapter, we provide a thorough review of several classical notions of connectivity on topological spaces and graphs, both in the ordinary and fuzzy sense. Basic results, which are easy or can be found in standard textbooks, are given without proof.

3.1 Topological Connectivity

In this section, we review the notions of connectivity and path-connectivity of ordinary topological spaces. For more details, the reader is referred to [24, 60].

3.1.1 Definition. A topological space E is said to be *connected* if there does not exist a pair (U_1, U_2) of disjoint nonempty open subsets of E such that $E = U_1 \cup U_2$. △

This definition applies to subspaces of E via the subspace topology. Using the definitions of subspace topology and of a connected space, we arrive at the following useful characterization of topological connectivity.

3.1.2 Proposition. Let E be a topological space. A subset A of E is connected if there does not exist a pair (G_1, G_2) of open subsets of E such that

(i) $A \subseteq G_1 \cup G_2$,

(ii) $A \cap G_1 \neq \emptyset$ and $A \cap G_2 \neq \emptyset$,

(iii) $A \cap G_1 \cap G_2 = \emptyset$. □

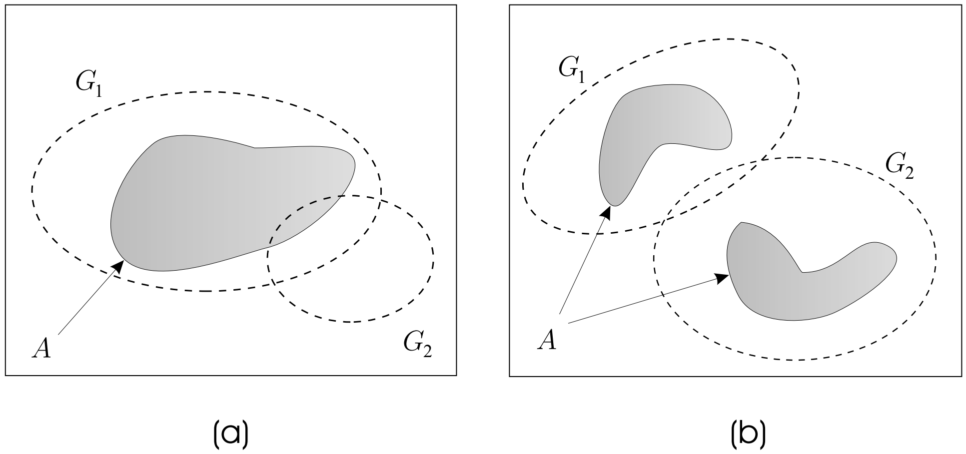


Figure 3.1: Topological connectivity on the 2-D Euclidean space: (a) Connected set. (b) Disconnected set.

Any pair (G_1, G_2) of open sets that satisfies conditions (i)–(iii) above is said to be a *separation* of A. Hence, A is connected if there does not exist a separation of A. Otherwise, it is said to be *disconnected*. Note that G_1 and G_2 are *not* required to be disjoint, as long as their intersection is disjoint from A, which is condition (iii). Note also that, according to this definition, the empty set and the points are always connected.

Given a subset A, its *connected components* are defined as the maximal connected subsets of A; i.e., C is a connected component of A if it is connected and there is no other connected subset of A that contains C. Of course, A is connected if and only if it contains a single connected component.

For instance, the interval $[0, 1]$ in the Euclidean line $\mathbb{R}$ is connected, whereas the union of intervals $[0, 1] \cup [2, 3]$ is not, since it consists of two connected components, $[0, 1]$ and $[2, 3]$. Fig. 3.1 depicts an example in the 2-D Euclidean space. The set A depicted in Fig. 3.1(a) is connected since it cannot be separated by any two open sets in E (the open sets G_1 and G_2 depicted in Fig. 3.1(a) satisfy (i) and (ii) in Proposition 3.1.2 but do not satisfy (iii)). On the other hand, the set A depicted in Fig. 3.1(b) is disconnected, since there exists a separation of A (the open sets G_1 and G_2 depicted in Fig. 3.1(b) satisfy (i)–(iii) in Proposition 3.1.2). Note that this set is comprised of two connected components.

The following proposition summarizes a few classical results on topological connectivity.

3.1.3 Proposition. Let E be a topological space.

(a) If subsets $\{A_\alpha\}$ of E are connected and $\bigcap A_\alpha \neq \emptyset$, then $\bigcup A_\alpha$ is connected.

(b) If A is a connected subset of E, then $\overline{A}$ is connected.

(c) The connected components $\{C_\alpha\}$ of a subset A of E are disjoint and $A = \bigcup C_\alpha$.

(d) The connected components of a closed subset of E are closed in E. □

Another definition of connectivity, based on topological spaces, is given below.

3.1.4 Definition. A topological space E is said to be *path-connected* if every two points in E can be joined by a *path*, i.e., a continuous map from $[0, 1]$ into E. △

This definition extends to subsets of E via the subspace topology. It is easy to see that a subset A of E is path-connected if, given two points in A, there is a path in A that joins them. For instance, $[0, 1]$ is path-connected, but $[0, 1] \cup [2, 3]$ is not. Note that the empty set and the points are path-connected. The *path-connected components* of a subset A of E are the maximally path-connected subsets of A.

Path-connectivity is a specialization of topological connectivity, as stated by the next result.

3.1.5 Proposition. Let E be a topological space.

(a) A subset A of E is path-connected if it is connected. The converse is not true in general.

(b) If E is the Euclidean space, then an *open* subset A of E is path-connected if and only if it is connected. □

The following result is the analog of Proposition 3.1.3 for path-connectivity.

3.1.6 Proposition. Let E be a topological space.

(a) If subsets $\{A_\alpha\}$ of E are path-connected and $\bigcap A_\alpha \neq \emptyset$, then $\bigcup A_\alpha$ is path-connected.

(b) The path-connected components $\{C_\alpha\}$ of a subset A of E are disjoint and $A = \bigcup C_\alpha$. □

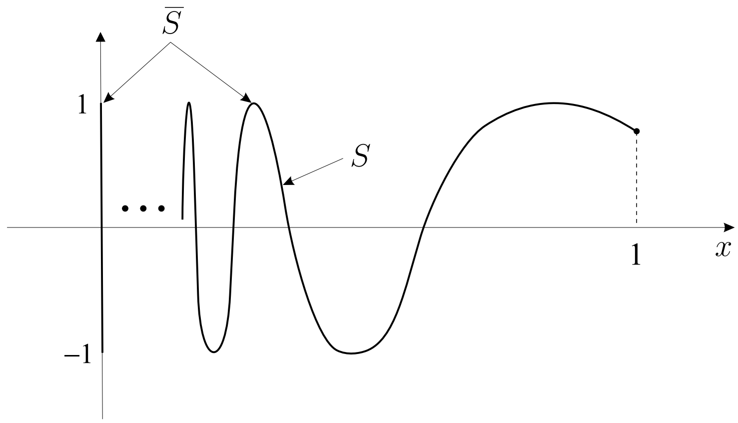


Figure 3.2: The Topologist's sine curve. The set $\overline{S}$ is connected, but not path-connected.

Note that A being path-connected does *not* imply that $\overline{A}$ is path-connected. Likewise, the path-connected components of a closed set need not be closed. A classical counterexample that shows these facts, and proves that connectivity does not in general imply path-connectivity, is known as the "Topologist's sine curve" (see Fig. 3.2). Consider the curve S in the 2-D Euclidean space given by $S = \{x \times \sin(1/x) \mid 0 < x \leq 1\}$. The Topologist's sine curve is the closure of this curve: $\overline{S} = (\{0\} \times [-1,1]) \cup S$. It can be shown that the Topologist's sine curve is connected, but not path-connected. Furthermore, S is path-connected, but $\overline{S}$ is not. In addition, the Topologist's sine curve is closed and its path-connected components are the sets $\{0\} \times [-1,1]$ and S, but S is not closed.

3.2 Graph-Theoretic Connectivity

In the case when E is a discrete space, connectivity is usually defined by means of a graph. In this section, we review the notions of connectivity and k-connectivity of (ordinary) graphs, as well as their "subspace" versions. For more details, the reader is referred to [22].

3.2.1 Definition. A *graph* is a pair $G = (V, L)$, where V is the set of *vertices* of G and $L \subseteq V \times V$ is the set of *edges* of G. △

It is allowed that $V = \emptyset$, in which case the resulting graph $G = \emptyset$ is referred to as the *null graph.* The number $|V|$ of vertices of G is the *order* of the graph. Note that $|V|$ may be infinite, and that $|V| = 0$ for the null graph. If $(v, w) \in L$, v and w are said to be *adjacent*, for $v, w \in V$. We assume that $(v, w) \in L \Rightarrow (w, v) \in L$, for $v, w \in V$; i.e., the graph $G = (V, L)$ is *undirected*.

If $G = (V, L)$ is a graph, a *path* in G between two given vertices $v_1, v_K \in V$ is a sequence $\Pi = \{v_1, v_2, \ldots, v_K\}$, where $v_i \in V$, such that $(v_i, v_{i+1}) \in L$, for $i = 1, 2, \ldots, K-1$. The definition of a connected graph is given below.

3.2.2 Definition. A graph $G = (V, L)$ is said to be *connected* if any two of its vertices are linked by a path in G. △

Recall that the definition of a connected topological space can be extended to subsets of the space via the concept of topological subspace. In similar fashion, the definition of a connected graph can be extended to subsets of the graph by means of the concept of the induced subgraph.

3.2.3 Definition. Let $G = (V, L)$ be a graph, and $U \subseteq V$ be a subset of the vertices of G. The graph $G[U] = (U, L')$, where $L' \subseteq L$ is the subset of the edges of G that link the vertices in U, is a graph known as the *subgraph induced* by U. △

This leads to the following definition.

3.2.4 Definition. Let $G = (V, L)$ be a graph. A subset $U \subseteq V$ of vertices of G is said to be *connected* in G if the induced subgraph $G[U]$ is connected. △

Hence, U is connected in G if any two of its vertices are linked by a path with vertices in U and edges in L. If U is not connected, it is said to be *disconnected*. Note that the empty set and the points are connected. The maximal connected subsets of U are called the *connected components* of U.

The following result is easy to prove.

3.2.5 Proposition. Let $G = (V, L)$ be a graph.

(a) If subsets $\{U_\alpha\}$ of V are connected and $\bigcap U_\alpha \neq \emptyset$, then $\bigcup U_\alpha$ is connected.

(b) The connected components $\{U_\alpha\}$ of a subset U of V are disjoint and $U = \bigcup U_\alpha$. □

Two cases of interest to image processing and analysis are obtained by taking the set of vertices to be points (m, n) in a subset of the two-dimensional discrete space $\mathbb{Z}^2$. Two vertices $v = (m, n)$ and $v' = (m', n')$ are said to be 4-*adjacent* if $|m - m'| + |n - n'| = 1$,

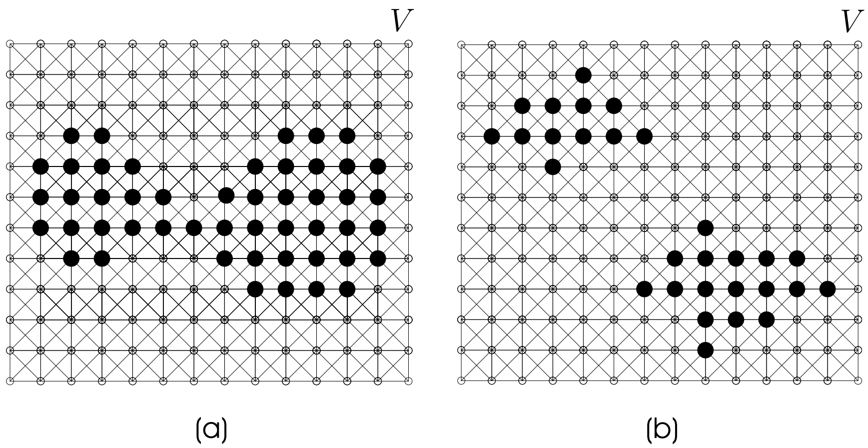


Figure 3.3: Graph-theoretic connectivity on $\mathbb{Z}^2$, assuming 8-adjacency connectivity: (a) Connected set. (b) Disconnected set.

whereas v and v' are said to be 8-*adjacent* if $\max\{|m - m'|, |n - n'|\} = 1$. This leads to the classical notions of 4- and 8-adjacency connectivity. Fig. 3.3 illustrates the concept of 8-adjacency connectivity. The set depicted in Fig. 3.3(a) is connected, since every pair of points is connected by a path with vertices in the set and edges in the underlying graph. On the other hand, the set depicted in Fig. 3.3(b) is disconnected. Note that this set is comprised of two connected components. In similar fashion, one can define 6-, 18- and 26-adjacency connectivity on the discrete space $\mathbb{Z}^3$, as well as higher-dimensional adjacency connectivities on $\mathbb{Z}^n$, for $n \geq 4$.

Another definition of graph-theoretic connectivity, which generalizes the definition discussed above, is given next.

3.2.6 Definition. Given a positive integer k, a graph $G = (V, L)$ is said to be *k-connected* if $G = \emptyset$, or $|V| \geq k$ and $G[V \setminus U]$ is connected for each $U \subseteq V$ such that $|U| < k$. △

In other words, a graph is k-connected if it is the null graph, or if one is able to delete any number of vertices less than k and still obtain a connected graph. It is obvious that, when $k = 1$, k-connectivity reduces trivially to graph-theoretic connectivity, given by Definition 3.2.2. In addition, it is clear that, if $k_1 \geq k_2$, then k_1-connectivity implies k_2-connectivity. This property means that k-connectivity defines a degree of connectivity for graphs, as given next.

3.2.7 Definition. Let $G = (V, L)$ be a graph. The value

$$\kappa = \sup \{k \in \mathbb{Z}_+ \mid G \text{ is } k\text{-connected}\} \tag{3.1}$$

is called the *degree of connectivity* of G. △

There is an alternative characterization of k-connectivity, given by the next proposition.

3.2.8 Proposition. A graph is k-connected if and only if any two vertices in the graph are linked by k non-intersecting paths. □

By using the concept of induced subgraphs, one is able to extend the definition of k-connectivity to subsets of vertices.

3.2.9 Definition. Given a graph $G = (V, L)$, a subset U of V is said to be k-connected in G if $U = \emptyset$, or $|U| \geq k$ and $U \setminus W$ is connected in G for all subsets $W \subseteq U$ such that $|W| < k$. △

As before, 1-connectivity reduces trivially to plain graph-theoretic connectivity and, if $k_1 \geq k_2$, then k_1-connectivity implies k_2-connectivity. In addition, the maximal k-connected subsets of U are called the *k-connected components* of U.

The notion of degree of connectivity also extends naturally to subsets of vertices, via the following definition.

3.2.10 Definition. Let $G = (V, L)$ be a graph, and consider the mapping $\kappa_G : \mathcal{P}(V) \to \mathbb{Z}_+$, given by

$$\kappa_G(U) = \sup \{k \in \mathbb{Z}_+ \mid U \text{ is } k\text{-connected in } G\}. \tag{3.2}$$

The quantity $\kappa_G(U)$ is the degree of connectivity of U in G. △

The following result is the extension of Proposition 3.2.5 to the case of k-connectivity.

3.2.11 Proposition. Let $G = (V, L)$ be a graph, and k be a positive integer.

(a) If subsets $\{U_\alpha\}$ of V are k-connected and $|\bigcap U_\alpha| \geq k$, then $\bigcup U_\alpha$ is k-connected.

(b) The k-connected components of a subset U of V overlap by at most $k-1$ vertices. □

Proof. (a): Suppose that $W \subseteq \bigcup U_\alpha$ with $|W| < k$. By definition of k-connectivity, we need to show that $(\bigcup U_\alpha) \setminus W$ is connected. Since U_α is assumed to be k-connected, $U_\alpha \setminus W$ is connected, for all α. Moreover, $\bigcap(U_\alpha \setminus W) = (\bigcap U_\alpha) \setminus W \neq \emptyset$, since $|\bigcap U_\alpha| \geq k$. Therefore, $(\bigcup U_\alpha) \setminus W = \bigcup(U_\alpha \setminus W)$ is connected.

(b): If two distinct k-connected components U_α and U_β of U overlap by k vertices or more, then it follows from part (a) that $U_\alpha \cup U_\beta$ is k-connected. In this case however neither U_α nor U_β are k-connected components of U, which is a contradiction. Therefore, U_α and U_β should overlap by at most $k-1$ vertices. Q.E.D.

When $k = 1$, the above result clearly reduces to Proposition 3.2.5. Note that, if $k > 1$, the union of the k-connected components of a set U does not necessarily equal U.

Fig. 3.4 provides an example, where $G = (\mathbb{Z}^2, L)$. The set depicted in Fig. 3.4(a) is 2-connected, since one can delete any point and obtain an 8-adjacency connected set or, equivalently, there are two non-intersecting paths between any two points. However, this set is clearly not 3-connected, since it is possible to delete two vertices and obtain a set that is not 8-connected. In particular, the degree of connectivity of this set is 2. On the other hand, the set depicted in Fig. 3.4(b) is 1-connected, or simply connected, but not 2-connected, because removing the indicated *cutvertex* [22] produces a set that is not connected, according to 8-adjacency connectivity. Equivalently, any two paths between a point on the left bulge and a point on the right bulge must join at the cutvertex, so it is not possible to find two non-intersecting paths linking these two points. In particular, the degree of connectivity of this set is 1. We mention that, in graph theory, 2-connected components are also called *blocks* [22].

3.3 Fuzzy Topological Connectivity

Several definitions of connectivity on fuzzy topological spaces have appeared in the literature [1, 25, 48, 51, 61, 63]. In this subsection, we examine two of them, which are both natural extensions of the notion of ordinary topological connectivity.

First, we define the concept of τ-connectivity for fuzzy topological spaces (this definition is similar to the one in [63, Defn. 3.1]). Recall that "$*$" denotes the negation associated with lattice $\mathcal{T}$.

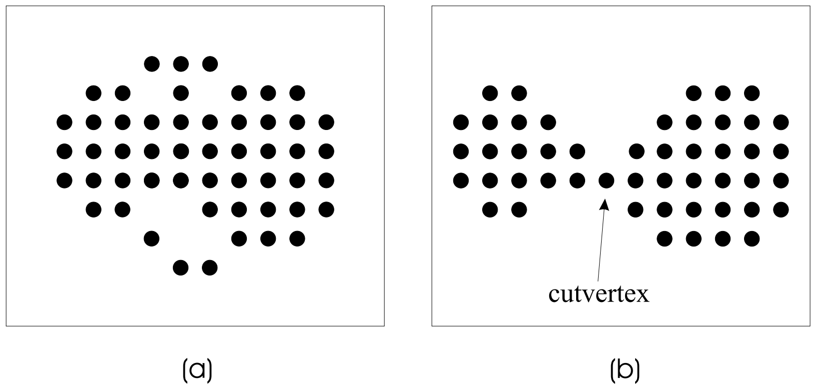


Figure 3.4: Graph-theoretic k-connectivity and degree of connectivity in $G = (\mathbb{Z}^2, L)$, assuming 8-adjacency connectivity: (a) A set that is 2-connected, but not 3-connected. Its degree of connectivity is 2. (b) A set that is connected, but not 2-connected. Its degree of connectivity is 1.

3.3.1 Definition. A $\mathcal{T}$-fuzzy topological space (E, Δ) is said to be τ-*connected*, for $\tau \in \mathcal{T} \setminus \{O\}$, if there does not exist a pair (μ_1, μ_2) of disjoint nonzero open fuzzy sets in Δ such that $(\mu_1 \vee \mu_2)(v) \not\leq \tau^*$, for all $v \in E$. If (E, Δ) is I-connected, it is said to be *fully connected*, whereas, if (E, Δ) is not τ-connected for any $\tau \in \mathcal{T} \setminus \{O\}$, it is said to be *fully disconnected*. △

Note that τ-connectivity defines a *degree of connectivity* for fuzzy topological spaces, in the sense that, if $\tau_1 \geq \tau_2$, then τ_1-connectivity implies τ_2-connectivity.

The following result gives a sufficient condition for a fuzzy topological space to be τ-connected.

3.3.2 Proposition. Let (E, Δ) be a $\mathcal{T}$-fuzzy topological space. For $\tau \in \mathcal{T} \setminus \{O\}$,

$$(E, Y_{\tau^*}(\Delta)) \text{ is connected } \Rightarrow (E, \Delta) \text{ is } \tau\text{-connected.} \tag{3.3}$$

□

Proof. We show the contrapositive of (3.3). Suppose that (E, Δ) is not τ-connected. Then, there exists a pair (μ_1, μ_2) of disjoint nonzero open fuzzy sets in Δ such that $(\mu_1 \vee \mu_2)(v) \not\leq \tau^*$, for all $v \in E$; i.e., $E = Y_{\tau^*}(\mu_1 \vee \mu_2) = Y_{\tau^*}(\mu_1) \cup Y_{\tau^*}(\mu_2)$. Since $Y_{\tau^*}(\mu_1)$ and $Y_{\tau^*}(\mu_2)$ are disjoint nonempty open sets in $(E, Y_{\tau^*}(\Delta))$, this space is disconnected. Q.E.D.

The condition given in Proposition 3.3.2 is not necessary, in general. For a counterexample, let $E = \{v_1, v_2\}$, $\mathcal{T} = \{0, 1, 2\}$, and $\Delta = \{\{0, 0\}, \{1, 1\}, \{2, 1\}, \{1, 2\}, \{2, 2\}\}$, where

$\{\tau_1, \tau_2\}$ denotes the fuzzy set given by $\mu(v_1) = \tau_1$, $\mu(v_2) = \tau_2$. It is easy to check that, with $\tau^* = 2 - \tau$, (E, Δ) is a $\mathcal{T}$-fuzzy topological space that is fully connected, since there is no pair (μ_1, μ_2) of disjoint nonzero open fuzzy sets in Δ such that $(\mu_1 \vee \mu_2)(v) > 0$, for all $v \in E$. In particular, (E, Δ) is 1-connected. However, $Y_{1^*}(\Delta) = Y_1(\Delta) = \{\emptyset, E, \{v_1\}, \{v_2\}\}$, so that $(E, Y_{1^*}(\Delta))$ is disconnected.

In the particular case of $\overline{\mathbb{R}}$-fuzzy topological spaces based on topology pyramids, discussed in Chapter 2, the condition in Proposition 3.3.2 is both necessary and sufficient. This is shown by the following proposition (note that in this case $\mathcal{T} = \overline{\mathbb{R}}$, with $\tau^* = -\tau$, for $\tau \in \overline{\mathbb{R}}$).

3.3.3 Proposition. Let $\mathbf{P} = \{\mathcal{G}_\tau \mid \tau \in \overline{\mathbb{R}} \setminus \{\infty\}\}$ be a topology pyramid on E, and let $(E, \Delta(\mathbf{P}))$ be the associated $\overline{\mathbb{R}}$-fuzzy topological space, where $\Delta(\mathbf{P})$ is given by (2.51). For $\tau \in \overline{\mathbb{R}} \setminus \{-\infty\}$,

$$(E, \Delta(\mathbf{P})) \text{ is } \tau\text{-connected} \quad \Leftrightarrow \quad (E, Y_{-\tau}(\Delta(\mathbf{P}))) = (E, \mathcal{G}_{-\tau}) \text{ is connected.} \tag{3.4}$$

□

Proof. The reverse implication follows from Proposition 3.3.2. We now establish the direct implication by showing its contrapositive. Given $\tau \in \overline{\mathbb{R}} \setminus \{-\infty\}$, suppose that $(E, \mathcal{G}_{-\tau})$ is disconnected. Then, there exists a pair (U_1, U_2) of disjoint nonempty open sets in $\mathcal{G}_{-\tau}$ such that $U_1 \cup U_2 = E$. Since $\mathbf{P}$ is a topology pyramid, we have $\mathcal{G}_{-\tau} = \bigcup_{s > -\tau} \mathcal{G}_s$, so that there exists a $\tau' > -\tau$ such that $U_1, U_2 \in \mathcal{G}_{\tau'}$. In addition, since $\mathcal{G}_{\tau'} \subseteq \mathcal{G}_s$, for all $s \leq \tau'$, we have that $U_1, U_2 \in \mathcal{G}_s$, for all $s \leq \tau'$. Now, consider the fuzzy sets $\mu_1, \mu_2 \in \overline{\mathbb{R}}^E$, given by

$$\mu_1(v) = \begin{cases} \tau', & \text{if } v \in U_1 \\ -\infty, & \text{otherwise} \end{cases} \quad \text{and} \quad \mu_2(v) = \begin{cases} \tau', & \text{if } v \in U_2 \\ -\infty, & \text{otherwise} \end{cases}. \tag{3.5}$$

It is clear that

$$Y_s(\mu_1) = \begin{cases} U_1, & \text{if } s < \tau' \\ \emptyset, & \text{otherwise} \end{cases} \quad \text{and} \quad Y_s(\mu_2) = \begin{cases} U_2, & \text{if } s < \tau' \\ \emptyset, & \text{otherwise} \end{cases}, \tag{3.6}$$

so that $Y_s(\mu_1), Y_s(\mu_2) \in \mathcal{G}_s$, for all $s \in \overline{\mathbb{R}} \setminus \{\infty\}$. From (2.51), it follows that $\mu_1, \mu_2 \in \Delta(\mathbf{P})$. But, clearly, μ_1, μ_2 are disjoint nonzero open fuzzy sets such that $(\mu_1 \vee \mu_2)(v) = \tau' \not\leq -\tau$, for all $v \in E$. Hence, $(E, \Delta(\mathbf{P}))$ is not τ-connected. Q.E.D.

As a consequence of the above result, a fuzzy topological space based on a topology pyramid $\mathbf{P} = \{\mathcal{G}_\tau \mid \tau \in \overline{\mathbb{R}} \setminus \{\infty\}\}$ is fully connected if and only if the *base* topological space $(E, \mathcal{G}_{-\infty})$ is connected.

The definition of τ-connectivity applies to subsets of E via the notion of fuzzy subspace topology. This leads to the following useful characterization (compare with Proposition 3.1.2). Recall that A denotes a fuzzy set that happens to be crisp.

3.3.4 Proposition. Let (E, Δ) be a $\mathcal{T}$-fuzzy topological space. For $\tau \in \mathcal{T} \setminus \{O\}$, a subset A of E is *τ-connected* if there does not exist a pair (μ_1, μ_2) of open fuzzy sets in (E, Δ) such that

(i) $(\mu_1 \vee \mu_2)(v) \not\lesssim \tau^*$, for $v \in A$,

(ii) $A \wedge \mu_1 \neq O$ and $A \wedge \mu_2 \neq O$,

(iii) $A \wedge \mu_1 \wedge \mu_2 = O$. □

Any pair (μ_1, μ_2) of open fuzzy sets that satisfies conditions *(i)*–*(iii)* above is said to be a *τ-separation* of A. Note that the empty set and the points in E can never be τ-separated, so they are τ-connected, for every $\tau \in \mathcal{T} \setminus \{O\}$. The *$\tau$-connected components* of a set $A \subseteq E$ are the maximal τ-connected subsets of A.

The next result is the subspace version of Proposition 3.3.3.

3.3.5 Proposition. Let $\mathbf{P} = \{\mathcal{G}_\tau \mid \tau \in \overline{\mathbb{R}} \setminus \{\infty\}\}$ be a topology pyramid on E, and let $(E, \Delta(\mathbf{P}))$ be the associated $\overline{\mathbb{R}}$-fuzzy topological space, where $\Delta(\mathbf{P})$ is given by (2.51). For a given subset $A \subseteq E$ and $\tau \in \overline{\mathbb{R}} \setminus \{-\infty\}$,

$$A \text{ is } \tau\text{-connected in } (E, \Delta(\mathbf{P})) \;\Leftrightarrow\; A \text{ is connected in } (E, \mathcal{G}_{-\tau}). \tag{3.7}$$

□

Proof. "$\Leftarrow$": We show the contrapositive. Given $\tau \in \overline{\mathbb{R}} \setminus \{-\infty\}$, suppose that A is not τ-connected in $(E, \Delta(\mathbf{P}))$; i.e., there exists a τ-separation (μ_1, μ_2) of A in $\Delta(\mathbf{P})$. Let $G_i = Y_{-\tau}(\mu_i)$, for $i = 1, 2$. We show that (G_1, G_2) is a separation of A in $\mathcal{G}_{-\tau}$. Since $\mu_1, \mu_2 \in \Delta(\mathbf{P})$, we have that $G_1, G_2 \in \mathcal{G}_{-\tau}$, by Proposition 2.5.6. The condition $(\mu_1 \vee \mu_2)(v) \not\lesssim -\tau$, for $v \in A$ is clearly equivalent to $A \subseteq Y_{-\tau}(\mu_1 \vee \mu_2) = Y_{-\tau}(\mu_1) \cup Y_{-\tau}(\mu_2) = G_1 \cup G_2$. Also, $A \wedge \mu_1 \wedge \mu_2 = O \Rightarrow \emptyset = Y_{-\tau}(A \wedge \mu_1 \wedge \mu_2) = Y_{-\tau}(A) \cap Y_{-\tau}(\mu_1) \cap Y_{-\tau}(\mu_2) = A \cap G_1 \cap G_2$. Finally, let $i \in \{1, 2\}$. Clearly, $A \wedge \mu_i \neq O$ implies that there is a $v \in A$ s.t. $\mu_i(v) > -\infty$. Since $(\mu_1 \vee \mu_2)(v) \not\lesssim -\tau$ and μ_1, μ_2 are disjoint over A, we conclude that $\mu_i(v) \not\lesssim -\tau$; i.e., $\emptyset \neq A \cap Y_{-\tau}(\mu_i) = A \cap G_i$. Hence, A is not connected in $(E, \mathcal{G}_{-\tau})$.

"⇒": We show the contrapositive. Given $\tau \in \overline{\mathbb{R}} \setminus \{-\infty\}$, suppose that A is not connected in $(E, \mathcal{G}_{-\tau})$; i.e., there exists a separation (G_1, G_2) of A in $\mathcal{G}_{-\tau}$. From property (ii) of topology pyramids, we have that $\mathcal{G}_{-\tau} = \bigcup_{s>-\tau} \mathcal{G}_s$, which implies that there exists a $\tau_0 > -\tau$ such that $G_1, G_2 \in \mathcal{G}_{\tau_0}$. In addition, since $\mathbf{P}$ is decreasing, we have that $G_1, G_2 \in \mathcal{G}_s$, for all $s < \tau_0$. Now, consider the fuzzy sets $\mu_1, \mu_2 \in \overline{\mathbb{R}}^E$, given by

$$\mu_1(v) = \begin{cases} \tau_0, & \text{if } v \in G_1 \\ -\infty, & \text{otherwise} \end{cases} \quad \text{and} \quad \mu_2(v) = \begin{cases} \tau_0, & \text{if } v \in G_2 \\ -\infty, & \text{otherwise} \end{cases}, \tag{3.8}$$

for $v \in E$. We show that (μ_1, μ_2) is a τ-separation of A in $\Delta(\mathbf{P})$. It is clear that

$$Y_s(\mu_1) = \begin{cases} G_1, & \text{if } s < \tau_0 \\ \emptyset, & \text{otherwise} \end{cases} \quad \text{and} \quad Y_s(\mu_2) = \begin{cases} G_2, & \text{if } s < \tau_0 \\ \emptyset, & \text{otherwise} \end{cases}, \tag{3.9}$$

so that $Y_s(\mu_1), Y_s(\mu_2) \in \mathcal{G}_s$ for all $s \in \overline{\mathbb{R}} \setminus \{\infty\}$; i.e., $\mu_1, \mu_2 \in \Delta(\mathbf{P})$. Note that $A \subseteq G_1 \cup G_2$ implies that $(\mu_1 \vee \mu_2)(v) = \tau_0 \not\leq -\tau$, for all $v \in A$. Also, $A \cap G_i \neq \emptyset$ implies that there exists a $v \in A$ such that $\mu_i(v) = \tau_0 > -\infty$; i.e., $\mu_i \wedge A \neq O$, for $i = 1, 2$. Finally, $A \cap G_1 \cap G_2 = \emptyset$ implies that there does not exist $v \in A$ such that $v \in G_1$ and $v \in G_2$; it follows that $A \wedge \mu_1 \wedge \mu_2 = O$. Hence, A is not τ-connected in $(E, \Delta(\mathbf{P}))$. Q.E.D.

The following result is the fuzzy analog of Proposition 3.1.3.

3.3.6 Proposition. Let (E, Δ) be a $\mathcal{T}$-fuzzy topological space. For $\tau \in \mathcal{T} \setminus \{O\}$:

(a) If subsets $\{A_\alpha\}$ of E are τ-connected and $\bigcap A_\alpha \neq \emptyset$, then $\bigcup A_\alpha$ is τ-connected.

(b) The τ-connected components $\{C_\alpha\}$ of a subset A of E are disjoint and $A = \bigcup C_\alpha$. □

Proof. (a): Suppose that $\bigcup A_\alpha$ is not τ-connected. Then, we can find a τ-separation of $\bigcup A_\alpha$ into fuzzy open sets μ_1 and μ_2. Pick a point $v \in \bigcap A_\alpha$. We have $v \in \bigcup A_\alpha$, so that we must have either $v \wedge \mu_1 \neq O$ or $v \wedge \mu_2 \neq O$, but not both, where v denotes the fuzzy set associated with a crisp point $v \in E$. Assume that $v \wedge \mu_1 \neq O$. This implies, since $v \in \bigcap A_\alpha$, that $A_\alpha \wedge \mu_1 \neq O$, for all α. Since each set A_α is τ-connected, we must have $A_\alpha \wedge \mu_2 = O$, for all α. But this implies that $(\bigcup A_\alpha) \wedge \mu_2 = \bigcup (A_\alpha \wedge \mu_2) = O$, by the infinite distributivity of $\mathcal{T}$, which contradicts the assumption that μ_1, μ_2 is a τ-separation of $\bigcup A_\alpha$.

(b): Let C_1, C_2 be two τ-connected components of A such that $C_1 \cap C_2 \neq \emptyset$. From part (a), we have that $C = C_1 \cup C_2$ is τ-connected, which implies, by the maximality of C_1, C_2, that $C \subseteq C_1$ and $C \subseteq C_2$, that is, $C_1 = C_2$. Hence, all τ-connected components

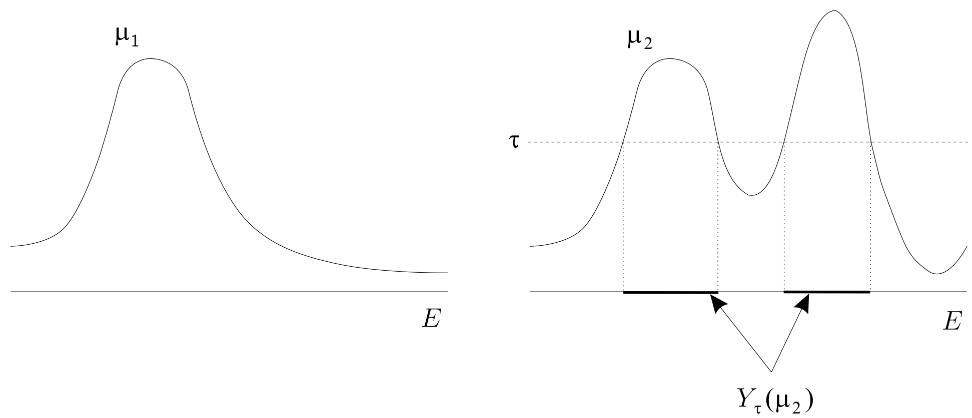


Figure 3.5: An example of level connectivity. The underlying fuzzy topological space is the ordinary Euclidean line. The fuzzy set μ_1 is level connected but the fuzzy set μ_2 is not, since $Y_\tau(\mu_2)$ is not connected.

of A are disjoint. Now, for $v \in A$, we have that either $\{v\}$ is a τ-connected component of A or $v \in C_\alpha$, for some τ-connected component C_α of A. Hence, $A = \bigcup_{v \in A} v \subseteq \bigcup C_\alpha \Rightarrow A = \bigcup C_\alpha$. Q.E.D.

Another type of fuzzy connectivity, which applies to fuzzy subsets of E, and thus to grayscale images defined on E, is defined next (this definition is similar to the one in [48, Defn. 2.6]).

3.3.7 Definition. Let (E, Δ) be a $\mathcal{T}$-fuzzy topological space. A fuzzy set $\mu \in \mathcal{T}^E$ is said to be *level connected* if the level sets $Y_\tau(\mu)$ are connected in the topological space $(E, Y_\tau(\Delta))$, for all $\tau \in \mathcal{T} \setminus \{I\}$. Otherwise, μ is said to be *level disconnected*. △

Clearly, this definition applies to crisp subsets of E as well. In particular, the space (E, Δ) is level connected if $(E, Y_\tau(\Delta))$ is connected, for all $\tau \in \mathcal{T} \setminus \{I\}$. Fig. 3.5 illustrates level connectivity, where the fuzzy topological space is taken to be the ordinary Euclidean line.

Note that the empty set and the fuzzy points in $\mathcal{T}^E$ are level connected. The *level connected components* of a fuzzy set $\mu \in \mathcal{T}^E$ are the maximal level connected fuzzy subsets of μ.

3.3.8 Definition. Given a family $\{\mu_\alpha\}$ of fuzzy sets in $\mathcal{T}^E$, we say that $\{\mu_\alpha\}$ is *overlapping* if $\bigcap_\alpha \{Y_\tau(\mu_\alpha) \mid Y_\tau(\mu_\alpha) \neq \emptyset\} \neq \emptyset$, for every $\tau \in \mathcal{T} \setminus \{I\}$. △

Note that, if $\{A_\alpha\}$ is a family of subsets of E, then $\{A_\alpha\}$ is overlapping if and only if $\bigcap A_\alpha \neq \emptyset$. Hence, the notion of overlapping for fuzzy sets is a generalization of the ordinary notion of set overlapping.

3.3.9 Proposition. Let (E, Δ) be a fuzzy topological space.

(a) If $\{\mu_\alpha\}$ is an overlapping family of level connected fuzzy sets in $\mathcal{T}^E$, then $\bigvee \mu_\alpha$ is level connected.

(b) The level connected components $\{\mu_\alpha\}$ of a fuzzy set $\mu \in \mathcal{T}^E$ are non-overlapping and $\mu = \bigvee \mu_\alpha$. □

PROOF. (a): Let $\tau \in \mathcal{T} \setminus \{I\}$. Note that $Y_\tau(\bigvee \mu_\alpha) = \bigcup_\alpha Y_\tau(\mu_\alpha) = \bigcup_\alpha \{Y_\tau(\mu_\alpha) \mid Y_\tau(\mu_\alpha) \neq \emptyset\}$. But each $Y_\tau(\mu_\alpha)$ is connected in $(E, Y_\tau(\Delta))$, since μ_α is level connected, and $\bigcap_\alpha \{Y_\tau(\mu_\alpha) \mid Y_\tau(\mu_\alpha) \neq \emptyset\} \neq \emptyset$, since $\{\mu_\alpha\}$ is overlapping. Hence, $Y_\tau(\bigvee \mu_\alpha)$ is connected in $(E, Y_\tau(\Delta))$, for all $\tau \in \mathcal{T} \setminus \{I\}$, so that $\bigvee \mu_\alpha$ is level connected.

(b): If μ_1, μ_2 are two overlapping level connected components of μ then, from part (a), $\mu_1 \vee \mu_2$ is level connected, so that $\mu_1 = \mu_2$, by the maximality of μ_1, μ_2. Hence, all level connected components of μ are non-overlapping. Now, for a fuzzy point $v_\tau \leq \mu$, either v_τ is a level connected component of μ or $v_\tau \leq \mu_\tau$, for some level connected component μ_α of μ. Hence, $\mu = \bigvee_{v_\tau \leq \mu} v_\tau \leq \bigvee \mu_\alpha \Rightarrow \mu = \bigvee \mu_\alpha$. Q.E.D.

Both definitions of fuzzy connectivity presented above provide a natural extension of the notion of topological connectivity, as the next result shows. The proof of this result is straightforward.

3.3.10 Proposition. Let $(E, \mathcal{G})$ be a topological space. Then,

$$(E, \mathcal{G}) \text{ is connected} \Leftrightarrow (E, \mathcal{G}) \text{ is fully connected} \Leftrightarrow (E, \mathcal{G}) \text{ is level connected.}$$

In addition,

$$(E, \mathcal{G}) \text{ is disconnected} \Leftrightarrow (E, \mathcal{G}) \text{ is fully disconnected} \Leftrightarrow (E, \mathcal{G}) \text{ is level disconnected.}$$

□

The relationship between the two previous definitions of fuzzy connectivity is given by the following proposition.

3.3.11 Proposition. Let (E, Δ) be a $\mathcal{T}$-fuzzy topological space. Then,

$$(E, \Delta) \text{ is level connected } \Rightarrow (E, \Delta) \text{ is fully connected.} \tag{3.10}$$

□

PROOF. This is a direct consequence of Proposition 3.3.2, by setting $\tau = I$. Q.E.D.

Note that the fuzzy topological space discussed in the counterexample after Proposition 3.3.2 is fully connected, but not level connected. This shows that the converse to (3.10) is not in general true.

The next result shows that, in the case of $\overline{\mathbb{R}}$-fuzzy topological spaces based on topology pyramids, the implication in Proposition 3.3.11 becomes an equivalence.

3.3.12 Proposition. Let $\mathbf{P} = \{\mathcal{G}_\tau \mid \tau \in \overline{\mathbb{R}} \setminus \{\infty\}\}$ be a topology pyramid on E, and let $(E, \Delta(\mathbf{P}))$ be the associated $\overline{\mathbb{R}}$-fuzzy topological space, where $\Delta(\mathbf{P})$ is given by (2.49). Then,

$$(E, \Delta(\mathbf{P})) \text{ is level connected } \Leftrightarrow (E, \Delta(\mathbf{P})) \text{ is fully connected.} \tag{3.11}$$

□

PROOF. The direct implication follows from Proposition 3.3.11. To show the reverse implication, note that, as a consequence of Proposition 3.3.3, the base topological space $(E, \mathcal{G}_{-\infty})$ is connected. But it can be easily verified that $\mathcal{G}_\tau \subseteq \mathcal{G}_{-\infty}$ implies that the space $(E, \mathcal{G}_\tau)$ must also be connected, for all $\tau \in \overline{\mathbb{R}} \setminus \{\infty\}$. Hence, $(E, \Delta(\mathbf{P}))$ is level connected. Q.E.D.

3.4 Fuzzy Graph-Theoretic Connectivity

Fuzzy topological connectivity retains the undesirable aspect of ordinary topological connectivity of not being adequate for dealing with discrete spaces. In this case, fuzzy connectivity can be best studied in the framework of *$\mathcal{T}$-fuzzy graphs*. The following is the $\mathcal{T}$-fuzzy analog of the original definition of a fuzzy graph [67, 95].

3.4.1 Definition. A *$\mathcal{T}$-fuzzy graph* is a pair $G = (V, \sigma)$, where V is the set of vertices and σ is a $\mathcal{T}$-fuzzy set on $V \times V$, called the *fuzzy edge set* of G. △

When $V = \emptyset$, the corresponding graph is a *null $\mathcal{T}$-fuzzy graph*. If σ is a crisp subset of $V \times V$, then G is a graph in the sense of Definition 3.2.1, so that ordinary graphs are special cases of $\mathcal{T}$-fuzzy graphs. Note that a $\mathcal{T}$-fuzzy graph can be thought of as a *weighted graph*; i.e., an ordinary graph with weights assigned to the edges.

The fuzzy edge set σ defines a binary *fuzzy relation* on V, a notion that dates back to Zadeh's original paper on fuzzy sets [96]. For two vertices $v, w \in V$, the quantity $\sigma(v, w)$ indicates the *degree of adjacency*, or *strength of connection*, between v and w. In what follows, we assume that the edge relation σ is *reflexive* (i.e., $\sigma(v, v) = I$, for $v \in V$) and *symmetric* (i.e., $\sigma(v, w) = \sigma(w, v)$, for $v, w \in V$). The symmetry requirement implies that the $\mathcal{T}$-fuzzy graph G is *undirected*.

With each $\mathcal{T}$-fuzzy graph G, we can associate a family of ordinary graphs $G_\tau = (V, X_\tau(\sigma))$, for $\tau \in \mathcal{T} \setminus \{O\}$, where $X_\tau(\sigma) = \{(v, w) \in V \times V \mid \sigma(v, w) \geq \tau\}$. The graphs G_τ are called the *τ-level graphs* associated with G. It is clear that, if $\tau_1 \geq \tau_2$, then G_{τ_1} is a subgraph of G_{τ_2}. It will become clear below that the τ-level graphs of a $\mathcal{T}$-fuzzy graph play, to some extent, the same role as the τ-level topological spaces of a fuzzy topological space.

The following is the fuzzy analog of the concept of an induced subgraph (see Definition 3.2.3).

3.4.2 Definition. Let $G = (V, \sigma)$ be a $\mathcal{T}$-fuzzy graph, and let $U \subseteq V$ be a subset of the vertices of G. The $\mathcal{T}$-fuzzy graph $G[U] = (U, \sigma')$, where

$$\sigma'(v, w) = \begin{cases} \sigma(v, w), & \text{if } v, w \in U \\ O, & \text{otherwise} \end{cases}, \tag{3.12}$$

is called the *$\mathcal{T}$-fuzzy subgraph induced by U.* △

Note that, if G is an ordinary graph (i.e., if σ is a ordinary subset of $V \times V$), then the previous definition agrees with Definition 3.2.3.

Given a $\mathcal{T}$-fuzzy graph $G = (V, \sigma)$, a *fuzzy path* in G, between two given vertices $v_1, v_K \in V$, is a sequence $\Pi = \{v_1, v_2, \ldots, v_K\}$, where $v_i \in V$, such that $\sigma(v_i, v_{i+1}) > O$, for $i = 1, 2, \ldots, K-1$. Clearly, a path in an ordinary graph is a special case of a fuzzy path. The *strength* $s(\Pi)$ of a fuzzy path $\Pi = \{v_1, v_2, \ldots, v_K\}$ is defined as $s(\Pi) = \bigwedge_{i=1}^{K-1} \sigma(v_i, v_{i+1})$. The set of all fuzzy paths between two vertices $v, w \in V$ is denoted by Π_{vw}.

3.4.3 Definition. Let $G = (V, \sigma)$ be a $\mathcal{T}$-fuzzy graph. The *degree of connectivity* $c(v, w)$ between two vertices $v, w \in V$ is defined by

$$c(v, w) = \bigvee_{\Pi \in \Pi_{vw}} s(\Pi) = \bigvee_{\Pi \in \Pi_{vw}} \bigwedge_{i=1}^{K-1} \{\sigma(v_i, v_{i+1}) \mid v_i, v_{i+1} \in \Pi\}. \tag{3.13}$$

△

To simplify analysis, we assume from this point on that the vertex set V is finite and that $\mathcal{T}$ is a chain. In this case, the strength of a path corresponds to the "weakest link" between any vertices in the path, and the degree of connectivity between two vertices corresponds to the strength of the "best path" between the vertices, since all the suprema and infima involved are achieved.

The following may be considered to be the discrete analog of fuzzy topological τ-connectivity (see Definition 3.3.1). The definition below coincides with the notion of *strong* τ*-connectivity*, which appears in [95].

3.4.4 Definition. A $\mathcal{T}$-fuzzy graph $G = (V, \sigma)$ is said to be τ*-connected*, for $\tau \in \mathcal{T} \setminus \{O\}$, if $c(v, w) \geq \tau$, for all $v, w \in V$. If G is I-connected, it is said to be *fully connected*, whereas, if G is not τ-connected, for any $\tau \in \mathcal{T} \setminus \{O\}$, it is said to be *fully disconnected.* △

In other words, a $\mathcal{T}$-fuzzy graph is τ-connected if the degree of connectivity between any pair of vertices is at least τ or, equivalently, if there exists a fuzzy path of strength at least τ between any pair of vertices. Note that a $\mathcal{T}$-fuzzy graph is fully connected if and only if it is an ordinary connected graph. Note also that τ-connectivity defines a degree of connectivity for $\mathcal{T}$-fuzzy graphs, in the sense that, if $\tau_1 \geq \tau_2$, then τ_1-connectivity implies τ_2-connectivity.

The following result characterizes τ-connectivity of a $\mathcal{T}$-fuzzy graph in terms of its τ-level graphs.

3.4.5 Proposition. Let $G = (V, \sigma)$ be a $\mathcal{T}$-fuzzy graph. For $\tau \in \mathcal{T} \setminus \{O\}$,

$$G = (V, \sigma) \text{ is } \tau\text{-connected} \quad \Leftrightarrow \quad G_\tau = (V, X_\tau(\sigma)) \text{ is connected.} \tag{3.14}$$

□

Proof. Let v, w be any two given vertices in V. As we argued before, G is τ-connected if and only if there exists a fuzzy path $\Pi = \{v = v_1, v_2, \ldots, v_K = w\}$ such that $s(\Pi) \geq \tau$; i.e., such that $\sigma(v_i, v_{i+1}) \geq \tau$, for $i = 1, 2, \ldots, K-1$. But this is true if and only if Π is a path in $G_\tau = (V, X_\tau(\sigma))$; i.e., if and only if G_τ is connected. Q.E.D.

The definition of τ-connectivity can be applied to subsets of V via the notion of induced $\mathcal{T}$-fuzzy subgraphs.

3.4.6 Definition. Let $G = (V, \sigma)$ be a $\mathcal{T}$-fuzzy graph. For $\tau \in \mathcal{T} \setminus \{O\}$, a subset $U \subseteq V$ of vertices of G is said to be *τ-connected* if the induced $\mathcal{T}$-fuzzy subgraph $G[U]$ is τ-connected. △

Note that the empty set and the points are (voidly) τ-connected, for all $\tau \in \mathcal{T} \setminus \{O\}$. The maximal τ-connected subsets of U are called the *τ-connected components* of U (or the *τ-clusters* of U, in the terminology of [95]).

The next result is the "subspace" version of Proposition 3.4.5.

3.4.7 Proposition. Let $G = (V, \sigma)$ be a $\mathcal{T}$-fuzzy graph. For a given subset $U \subseteq V$ and $\tau \in \mathcal{T} \setminus \{O\}$,

$$U \text{ is } \tau\text{-connected in } G = (V, \sigma) \;\Leftrightarrow\; U \text{ is connected in } G_\tau = (V, X_\tau(\sigma)). \tag{3.15}$$

□

Proof. Let v, w be any two given vertices in U. The subset U is τ-connected in $G = (V, \sigma)$ if and only if there is a path $\Pi = \{v = v_1, v_2, \ldots, v_K = w\} \subseteq U$ such that $s(\Pi) \geq \tau$; i.e., such that $\sigma(v_i, v_{i+1}) \geq \tau$, for $i = 1, 2, \ldots, K-1$. But this is true if and only if Π is a path in U according to $G_\tau = (V, X_\tau(\sigma))$; i.e., if and only if U is connected in G_τ. Q.E.D.

The following result is the fuzzy analog of Proposition 3.2.5 (it follows directly from Propositions 3.2.5 and 3.4.7).

3.4.8 Proposition. Let $G = (V, \sigma)$ be a $\mathcal{T}$-fuzzy graph. For $\tau \in \mathcal{T} \setminus \{O\}$:

(a) If subsets $\{U_\alpha\}$ of V are τ-connected and $\bigcap U_\alpha \neq \emptyset$, then $\bigcup U_\alpha$ is τ-connected.

(b) The τ-connected components $\{U_\alpha\}$ of a subset U of V are disjoint and $U = \bigcup U_\alpha$. □

The usefulness of τ-connectivity of $\mathcal{T}$-fuzzy graphs in image analysis applications has been demonstrated, for instance, in [95], where it was used for hierarchical cluster analysis, or in [86], which proposes a scheme for defining fuzzy objects in grayscale images that corresponds to τ-connectivity of a $\mathcal{T}$-fuzzy graph, where the fuzzy edge relation $\sigma(v, w)$ is defined in terms of spatial adjacency and homogeneity of the grayscale values at v, w.

We now present a different type of fuzzy graph-theoretical connectivity, which has been proposed by A. Rosenfeld in [68–70]. This may be thought of as the discrete analog of the notion of fuzzy topological level connectivity (see Definition 3.3.7).

Consider the lattice of grayscale images $\mathcal{L} = \text{Fun}(E, \mathcal{T})$, where E is a finite subset of $\mathbb{Z}^n$ and $\mathcal{T}$ is a finite chain (e.g., $\mathcal{T} = \{0, 1, \ldots, R-1\}$, where $R \geq 2$ is a finite integer). We also assume an underlying graph $G = (E, L)$, which provides a connectivity on the space E. We have the following definition.

3.4.9 Definition. For each image $f \in \text{Fun}(E, \mathcal{T})$, we define the *topographic $\mathcal{T}$-fuzzy graph* $G^f = (E, \sigma_f)$, where

$$\sigma_f(v, w) = \begin{cases} f(v) \wedge f(w), & \text{if } v \text{ and } w \text{ are adjacent} \\ O, & \text{otherwise} \end{cases}, \tag{3.16}$$

for $v, w \in E$. △

It is easy to see that the degree of connectivity between two points v, w in E, according to G^f, is given by

$$c_f(v, w) = \bigvee_{\Pi \in \Pi_{vw}} \bigwedge_{i=1}^{K-1} \{f(v_i) \mid v_i \in \Pi\}, \tag{3.17}$$

for $v, w \in E$, where Π_{vw} denotes the set of paths from v to w, according to the underlying graph $G = (E, L)$. Hence, the degree of connectivity between v and w is given by the least grayscale value along the "best" path from v to w. It follows that $c_f(v, w) \leq f(v) \wedge f(w)$, for all $v, w \in E$. If this upper bound is reached (i.e., if $c_f(v, w) = f(v) \wedge f(w)$), then we say that v is *topographically connected* to w [67]. Clearly, v and w are topographically connected if and only if there is a path from v to w along which the value of f never "dips" below the values of both $f(v)$ and $f(w)$.

The following definition introduces an interesting notion of connectivity for grayscale images based on the idea of topographic connectivity [69, 70].

3.4.10 Definition. An image $f \in \text{Fun}(E, \mathcal{T})$ is said to be *topographically connected* if every pair of points $v, w \in E$ is topographically connected. △

Note that the zero image and the pulses in $\text{Fun}(E, \mathcal{T})$ are topographically connected. The *topographically connected components* of an image $f \in \text{Fun}(E, \mathcal{T})$ are the maximal topographically connected images below f.

The following result characterizes topographic connectivity of an image $f \in \text{Fun}(E, \mathcal{T})$ in terms of the τ-level graphs of the associated topographic $\mathcal{T}$-fuzzy graph G^f.

3.4.11 Proposition. Let $f \in \mathrm{Fun}(E, \mathcal{T})$ and G^f be the associated topographic $\mathcal{T}$-fuzzy graph, given by Definition 3.4.9. The image f is topographically connected if and only if $X_\tau(f)$ is connected in the τ-level graph $G^f_\tau = (E, X_\tau(\sigma_f))$, for all $\tau \in \mathcal{T} \setminus \{O\}$. □

Proof. "⇒": Let $\tau \in \mathcal{T} \setminus \{O\}$ and $U = X_\tau(f)$. To show that U is connected in $G^f_\tau = (E, X_\tau(\sigma_f))$, we need to show that there is a path in $G^f_\tau[U]$ between any two vertices $v, w \in U$. It is easy to see that this is equivalent to finding a path $\Pi = \{v = v_1, v_2, \ldots, v_K = w\} \in \Pi_{v_1 v_K}$ such that $f(v_i) \geq \tau$, for $i = 1, 2, \ldots, K$. Suppose that there is no such path. Then, it easily follows from (3.17) that $c_f(v, w) < \tau$. But $f(v), f(w) \geq \tau$, so that $c_f(v, w) < f(v) \wedge f(w)$, contradicting the hypothesis that v and w are topographically connected.

"⇐": Let $v, w \in E$. We need to show that v and w are topographically connected; i.e., that $c_f(v, w) = f(v) \wedge f(w)$. If $f(v) = O$ or $f(w) = O$, then clearly $c_f(v, w) = O$, and we are done. Otherwise, let $\tau = f(v) \wedge f(w) \neq O$. We have that $v, w \in X_\tau(f)$, which is connected in $G^f_\tau = (E, X_\tau(\sigma_f))$. It follows that there is a path $\Pi = \{v = v_1, v_2, \ldots, v_K = w\} \in \Pi_{v_1 v_K}$ such that $f(v_i) \geq \tau$, for $i = 1, 2, \ldots, K$. Hence, $c_f(v, w) \geq \tau$. But, as we argued before, $c_f(v, w) \leq \tau$. It then follows that $c_f(v, w) = \tau = f(v) \wedge f(w)$, as required. Q.E.D.

By comparing this result with Definition 3.3.7, we see that the fuzzy graph-theoretic notion of topographic connectivity is indeed the discrete analog of the fuzzy topological notion of level connectivity. In particular, the example depicted in Fig. 3.5 can be easily discretized, which leads to an example of 1-D topographic connectivity.

Drawing on this parallel, we have the following definition (compare with Definition 3.3.8).

3.4.12 Definition. Given a family $\{f_\alpha\}$ of images in $\mathrm{Fun}(E, \mathcal{T})$, we say that $\{f_\alpha\}$ is *overlapping* if $\bigcap_\alpha \{X_\tau(f_\alpha) \mid X_\tau(f_\alpha) \neq \emptyset\} \neq \emptyset$, for every $\tau \in \mathcal{T} \setminus \{O\}$. △

We have the following result.

3.4.13 Proposition.

(a) If $\{f_\alpha\}$ is a family of overlapping topographically connected images in $\mathrm{Fun}(E, \mathcal{T})$, then $\bigvee f_\alpha$ is topographically connected.

(b) The topographic connected components $\{f_\alpha\}$ of an image $f \in \mathrm{Fun}(E, \mathcal{T})$ are non-overlapping and $f = \bigvee f_\alpha$. □

Proof. (a): Let $\tau \in \mathcal{T} \setminus \{O\}$. From the assumption that $\mathcal{T}$ is finite, we have that $X_\tau(\bigvee f_\alpha) = \bigcup_\alpha X_\tau(f_\alpha)$. From Proposition 3.4.11, $X_\tau(f_\alpha)$ is connected in $G_\tau^{f_\alpha}$, for each α. Since $G_\tau^{f_\alpha}$ is clearly a subgraph of $G_\tau^{\vee f_\alpha}$, it follows that $X_\tau(f_\alpha)$ is connected in $G_\tau^{\vee f_\alpha}$, for each α. Since $\bigcap_\alpha \{X_\tau(f_\alpha) \mid X_\tau(f_\alpha) \neq \emptyset\} \neq \emptyset$, it follows that $X_\tau(\bigvee f_\alpha) = \bigcup_\alpha X_\tau(f_\alpha) = \bigcup_\alpha \{X_\tau(f_\alpha) \mid X_\tau(f_\alpha) \neq \emptyset\}$ is connected in $G_\tau^{\vee f_\alpha}$. Since this holds for each $\tau \in \mathcal{T} \setminus \{O\}$, it follows from Proposition 3.4.11 that $\bigvee f_\alpha$ is topographically connected.

(b): If f_1 and f_2 are two overlapping topographically connected components of f then, from part (a), $f_1 \vee f_2$ is topographically connected, so that $f_1 = f_2$, by the maximality of f_1, f_2. Hence, all topographically connected components of f are non-overlapping. Now, for a pulse $\delta_{v,t} \leq f$, either $\delta_{v,t}$ is a topographically connected component of f, or $\delta_{v,t} \leq f_\alpha$, for some topographically connected component f_α of f. Hence, $f = \bigvee_{\delta_{v,t} \leq f} \delta_{v,t} \leq \bigvee f_\alpha \Rightarrow f = \bigvee f_\alpha$. Q.E.D.

The usefulness of topographic connectivity in image analysis applications has been demonstrated, for instance, in [21], where the concept of "intensity connectivity" is defined based on the notion of topographic connectivity.

Chapter 4

Connectivity on Complete Lattices

As we mentioned in Chapter 1, the standard notions of connectivity, such as topological connectivity and graph-theoretic connectivity, are incompatible. For example, although there is a topology on $\mathbb{Z}^2$ for which the topologically connected sets are the 4-adjacency connected sets in $\mathbb{Z}^2$, it can be shown that there exists no topology on $\mathbb{Z}^2$ that yields 8-adjacency connectivity. Furthermore, topological connectivity on Hausdorff spaces, such as the Euclidean space, cannot arise from graph-theoretic connectivity, since every pair of points is disconnected by definition. See [66] and the references therein for a more detailed discussion on these issues.

This state of affairs motivated G. Matheron and J. Serra to propose an axiomatic approach to binary connectivity, known as the theory of *connectivity classes*, which includes and unifies traditional concepts of connectivity, and allows the study of many interesting connectivity examples that are not covered by the classical definitions [77]. Recently, J. Serra showed how to extend the theory of connectivity classes to complete lattices, in a way that is consistent with the binary theory [78–80].

This chapter is organized as follows. Connectivity classes in complete lattices are discussed in Section 4.1, when we also study properties of connectivity openings and the reconstruction operator. In Section 4.2, we propose connectivity classes defined on ψ-invariant lattices, which include the classical notion of graph-theoretic k-connectivity and a new example of grayscale connectivity, the so-called flat grayscale connectivity. In Section 4.3, we study second-generation connectivities, including clustering-based connectivities and contraction-based connectivities. Finally, in Section 4.4, we investigate hyperconnectivity.

4.1 Fundamental Notions

In this section, we study the axiomatic formulation of connectivity classes in complete lattices, first proposed by G. Matheron and J. Serra. We also study the notions of connectivity openings and reconstruction, which are useful operators associated with a connectivity class. We present a classic result according to which a connectivity class can be equivalently specified in terms of connectivity openings, and provide a few novel results on semi-continuity properties of connectivity openings. Another original contribution is a result that shows that, in the case of infinite $\vee$-distributive lattices, a connectivity class can be equivalently specified in terms of a reconstruction operator.

4.1.1 Connectivity Classes

G. Matheron and J. Serra's original definition of a connectivity class [77] is given below.

4.1.1 Definition. Let E be a set. A family $\mathcal{C} \subseteq \mathcal{P}(E)$ is called a *connectivity class* in $\mathcal{P}(E)$ if the following conditions are satisfied:

(i) $\emptyset \in \mathcal{C}$,

(ii) $\{v\} \in \mathcal{C}$, for all $v \in E$,

(iii) for a family $\{C_\alpha\}$ in $\mathcal{C}$ such that $\bigcap C_\alpha \neq \emptyset$, we have that $\bigcup C_\alpha \in \mathcal{C}$.

The family $\mathcal{C}$ generates a *connectivity* on $\mathcal{P}(E)$, and the sets in $\mathcal{C}$ are said to be *connected*. $\triangle$

Axioms (i) and (ii) in Definition 4.1.1 require that the empty set and the points be always connected, whereas axiom (iii) requires that the union of intersecting connected sets be connected. This may be considered to be a minimal set of desirable requirements for connectivity. In particular, it is easy to see that these requirements imply that an object is partitioned by its connected components, which is a fundamental property of connectivity.

Definition 4.1.1 is applicable only to binary images. More recently, however, J. Serra extended this axiomatization to the general case of complete lattices [78–80]. In this dissertation, we consider the "canonical markers" axiomatization of connectivity classes, as defined in [79, 80]. According to this framework, one chooses a fixed sup-generating family in the lattice of interest to play the same role as the points in Definition 4.1.1. We have the following definition.

4.1.2 Definition. Let $\mathcal{L}$ be a lattice with a sup-generating family $\mathcal{S}$. A family $\mathcal{C} \subseteq \mathcal{L}$ is called a *connectivity class* in $\mathcal{L}$ if the following conditions are satisfied:

(*i*) $O \in \mathcal{C}$,

(*ii*) $\mathcal{S} \subseteq \mathcal{C}$,

(*iii*) for a family $\{C_\alpha\}$ in $\mathcal{C}$ such that $\bigwedge C_\alpha \neq O$, we have that $\bigvee C_\alpha \in \mathcal{C}$.

The family $\mathcal{C}$ generates a *connectivity* on $\mathcal{L}$, and the elements in $\mathcal{C}$ are said to be *connected*. △

Note that, when $\mathcal{L} = \mathcal{P}(E)$ with the points as sup-generators, the definition above coincides with Definition 4.1.1.

It will be useful to make some specializations. Following the terminology in [35], we say that a connectivity class is *strong* if the universal element I of $\mathcal{L}$ is connected; i.e., if $I \in \mathcal{C}$. In the case of the set lattice $\mathcal{P}(E)$, this means that $E \in \mathcal{C}$. In addition, if $E = \mathbb{R}^n$ or $E = \mathbb{Z}^n$, we say that a connectivity class $\mathcal{C}$ in $\mathcal{P}(E)$ is *translation-invariant* if $A \in \mathcal{C} \ \Leftrightarrow \ A_h \in \mathcal{C}$, for all $h \in E$.

Below, we give a few examples of connectivity classes (see also [35, 66, 79, 80]). Additional examples will be studied in Sections 4.2 and 4.3.

4.1.3 Example.

(a) (*Minimal connectivity class*). Consider a lattice $\mathcal{L}$ with sup-generating family $\mathcal{S}$. Suppose that the sup-generators in $\mathcal{S}$ satisfy the condition in axiom (*iii*) of Definition 4.1.2. Then, the family $\mathcal{C}_{\min} = \{O\} \cup \mathcal{S}$ is a connectivity class. In this case, only the least element and the sup-generators are connected.

(b) (*Maximal connectivity class*). Given a lattice $\mathcal{L}$, $\mathcal{C}_{\max} = \mathcal{L}$ is a connectivity class. In this case, every element in $\mathcal{L}$ is connected. This connectivity class is strong.

(c) (*Topological connectivity*). Let $(E, \mathcal{G})$ be a topological space, and let $\mathcal{L} = \mathcal{P}(E)$ with the points as sup-generators. It follows from the results in Section 3.1 that

$$\mathcal{C} = \{A \in \mathcal{P}(E) \mid A \text{ is connected in } (E, \mathcal{G})\}, \tag{4.1}$$

$$\mathcal{C}' = \{A \in \mathcal{P}(E) \mid A \text{ is path-connected in } (E, \mathcal{G})\} \tag{4.2}$$

are connectivity classes in $\mathcal{P}(E)$, such that $\mathcal{C}' \subseteq \mathcal{C}$. Note that, in the case of the Euclidean space $(\mathbb{R}^n, \mathcal{G}_e)$, both $\mathcal{C}$ and $\mathcal{C}'$ are strong and translation-invariant.

(d) (*Graph-theoretic connectivity*). Let $G = (V, L)$ be a graph, and let $\mathcal{L} = \mathcal{P}(V)$ with the points as sup-generators. It follows easily from the results in Section 3.2 that

$$\mathcal{C} = \{U \in \mathcal{P}(V) \mid U \text{ is connected in } G\} \tag{4.3}$$

is a connectivity class in $\mathcal{P}(V)$. For example, with $V = \mathbb{Z}^2$, this includes the case of 4- and 8-adjacency connectivities discussed in Section 3.2, in which case $\mathcal{C}$ is strong and translation-invariant.

(e) (*Fuzzy topological τ-connectivity*). Let (E, Δ) be a $\mathcal{T}$-fuzzy topological space, and let $\mathcal{L} = \mathcal{P}(E)$ with the points as sup-generators. It follows easily from the results in Section 3.3 that

$$\mathcal{C}_\tau = \{A \in \mathcal{P}(E) \mid A \text{ is } \tau\text{-connected in } (E, \Delta)\} \tag{4.4}$$

is a connectivity class in $\mathcal{P}(E)$, for each $\tau \in \mathcal{T} \setminus \{O\}$. Note that the connectivity class $\mathcal{C}$ in (4.1) is a special case of the present example.

(f) (*Fuzzy graph-theoretic τ-connectivity*). Let $G = (V, \sigma)$ be a $\mathcal{T}$-fuzzy graph, where V is finite and $\mathcal{T}$ is a chain, and let $\mathcal{L} = \mathcal{P}(V)$ with the points as sup-generators. It follows easily from the results in Section 3.4 that

$$\mathcal{C}_\tau = \{U \in \mathcal{P}(V) \mid U \text{ is } \tau\text{-connected in } G\} \tag{4.5}$$

is a connectivity class in $\mathcal{P}(V)$, for each $\tau \in \mathcal{T} \setminus \{O\}$. Note that example (d) is a special case of the present example.

(g) (*"Polygonal-line" connectivity*). Consider the lattice $\mathcal{L} = \mathcal{P}(\mathbb{R}^n)$ with the points as sup-generators. The family of all subsets C of $\mathbb{R}^n$ such that any two points of C can be joined by a polygonal line that lies entirely in C is clearly a connectivity class in $\mathcal{P}(\mathbb{R}^n)$. In this case, disconnected objects are subsets of $\mathbb{R}^n$ that have thin curved parts. This connectivity class is strong and translation-invariant. See Fig. 4.1 for an illustration. Note that the family of all subsets C of $\mathbb{R}^n$ such that any two points of C can be joined by a *straight* line that lies entirely in C (i.e., C is convex) *does not* form a connectivity class, since union of two intersecting convex sets may not be convex.

(h) (*"Support" connectivity*). Consider the function lattice $\mathcal{L} = \mathrm{Fun}(E, \mathcal{T})$, with the pulses as sup-generators. Assume that $\mathcal{P}(E)$ is furnished with a connectivity class $\mathcal{C}_E$. Then,

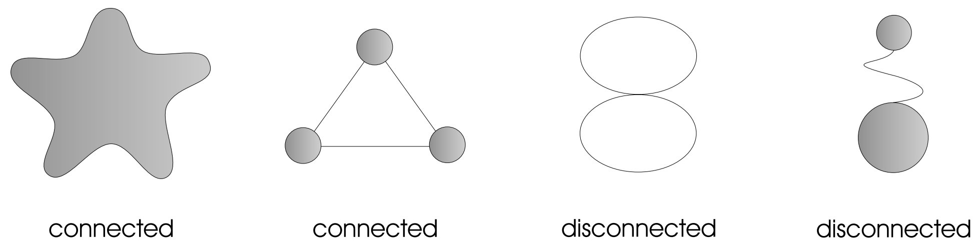


Figure 4.1: An illustration of "polygonal-line" connectivity.

a connectivity class $\mathcal{C}$ in $\text{Fun}(E, \mathcal{T})$ can be defined as the set of all functions that have connected support on E; i.e.,

$$\mathcal{C} = \{f \in \text{Fun}(E, \mathcal{T}) \mid \Omega(f) \in \mathcal{C}_E\}, \tag{4.6}$$

where $\Omega(f) = \{v \in E \mid f(v) > 0\}$ (here, "0" stands for the least element of lattice $\mathcal{T}$). Clearly, $\mathcal{C}$ is strong if $\mathcal{C}_E$ is. $\Diamond$

The notions of graph-theoretic k-connectivity, fuzzy topological level connectivity and fuzzy graph-theoretic topographic connectivity, discussed in Sections 3.2, 3.3 and 3.4, respectively, do not in general lead, in a straightforward way, to connectivity classes. This is mainly because the overlapping criterion in those cases cannot be expressed in terms of a simple pointwise infimum operation. In Section 4.4, these are shown to fit naturally in the framework of *hyperconnectivity*. Nevertheless, we show in Section 4.2 that, by defining suitable lattices, graph-theoretic k-connectivity can be formulated as a connectivity class.

Example 4.1.3(g) appears in [35]. The "support" connectivity of Example 4.1.3(h) is not very useful in practice, since, according to this connectivity, any image with strictly nonzero values will be connected, assuming that $\mathcal{C}_E$ is a strong connectivity class. This leads to a single connected component for the whole image, even if the image is composed of objects that should be viewed as separate connected components. As a matter of fact, the function lattice $\text{Fun}(E, \mathcal{T})$ is not really adequate for generating useful connectivity classes. The main difficulty lies in finding a family $\mathcal{C} \subseteq \text{Fun}(E, \mathcal{T})$ such that axiom (iii) of Definition 4.1.2 is satisfied. A solution to this problem is to define suitable lattices that allow the construction of interesting connectivity classes for functions. In Section 4.2.4, we use this approach to generate a potentially useful example of connectivity for grayscale images.

Let us now denote by $\text{Ccl}(\mathcal{L})$ the set of all connectivity classes in a lattice $\mathcal{L}$, with a fixed sup-generating family. It was shown in [79, Proposition 4] that $\text{Ccl}(\mathcal{L})$ is a complete lattice under the usual inclusion partial order relation for families of $\mathcal{L}$. The following is a more detailed version of that result.

4.1.4 Proposition. Let $\mathcal{L}$ be a lattice with a sup-generating family $\mathcal{S}$.

(a) The operator ϕ on $\mathcal{P}(\mathcal{L})$, given by

$$\phi(\mathcal{A}) = \bigcap\{\mathcal{C} \in \mathrm{Ccl}(\mathcal{L}) \mid \mathcal{C} \supseteq \mathcal{A}\}, \quad \mathcal{A} \in \mathcal{P}(\mathcal{L}), \tag{4.7}$$

is a closing on $\mathcal{P}(\mathcal{L})$, with invariance domain $\mathrm{Inv}(\phi) = \mathrm{Ccl}(\mathcal{L})$.

(b) $\mathrm{Ccl}(\mathcal{L})$ is an underlattice of $\mathcal{P}(\mathcal{L})$, with infimum $\bigcap \mathcal{C}_\alpha$ and supremum $\phi(\bigcup \mathcal{C}_\alpha)$. □

Proof. (a): Obviously, ϕ is increasing and extensive. To show that ϕ is idempotent, it suffices to show that $\phi(\mathcal{A}) \in \mathrm{Ccl}(\mathcal{L})$, for all $\mathcal{A} \in \mathcal{P}(\mathcal{L})$. Let $\mathbf{C} = \{\mathcal{C} \in \mathrm{Ccl}(\mathcal{L}) \mid \mathcal{C} \supseteq \mathcal{A}\}$. Since $\{O\} \in \mathcal{C}$ and $\mathcal{S} \subseteq \mathcal{C}$, for all $\mathcal{C} \in \mathbf{C}$, we have that $\{O\} \in \phi(\mathcal{A})$ and $\mathcal{S} \subseteq \phi(\mathcal{A})$, which shows axioms (i) and (ii) of connectivity classes. To show axiom (iii), consider a family $\{A_\alpha\} \subseteq \phi(\mathcal{A})$ such that $\bigwedge A_\alpha \neq O$. It follows from (4.7) that, for all $\mathcal{C} \in \mathbf{C}$, $\{A_\alpha\} \subseteq \mathcal{C} \Rightarrow \bigvee A_\alpha \in \mathcal{C}$, since $\mathcal{C}$ is a connectivity class. Therefore, $\bigvee A_\alpha \in \phi(\mathcal{A})$. Hence, ϕ is a closing. Now, if $\mathcal{A} \in \mathrm{Ccl}(\mathcal{L})$, it is clear from (4.7) that $\phi(\mathcal{A}) = \mathcal{A}$. Conversely, $\phi(\mathcal{A}) = \mathcal{A}$ implies that $\mathcal{A} \in \mathrm{Ccl}(\mathcal{L})$, since we have shown that $\phi(\mathcal{A}) \in \mathrm{Ccl}(\mathcal{L})$, for all $\mathcal{A} \in \mathcal{P}(\mathcal{L})$. Hence, $\mathrm{Inv}(\phi) = \mathrm{Ccl}(\mathcal{L})$.

(b): This follows directly from (a) and Proposition 2.2.2(a). Q.E.D.

The closing $\phi(\mathcal{A})$ gives the smallest connectivity class in $\mathcal{L}$ that contains the family $\mathcal{A} \in \mathcal{P}(\mathcal{L})$. Note that, as a direct consequence of part (b) of the previous proposition, the intersection of connectivity classes is a connectivity class as well.

4.1.2 Connectivity Openings

One of the most basic image analysis tasks that involves connectivity is extracting a connected component from an image marked with a marker. In the context of a complete lattice $\mathcal{L}$ with sup-generating family $\mathcal{S}$, it is natural to use as markers the sup-generators in $\mathcal{S}$. Given a sup-generator $x \in \mathcal{S}$, one would like to define an operator $\gamma_x(A)$ that extracts the connected component of A marked by x.

It turns out that connectivity classes are intimately related to such operators. Given a connectivity class $\mathcal{C}$ in $\mathcal{L}$, let

$$\gamma_x(A) = \bigvee\{C \in \mathcal{C} \mid x \leq C \leq A\}, \quad A \in \mathcal{L}, \tag{4.8}$$

for every $x \in \mathcal{S}$. This is known as the *connectivity opening* associated with the connectivity class $\mathcal{C}$ (it can be easily verified that γ_x is increasing, anti-extensive, and idempotent; hence, it is an opening). Note that axiom (*iii*) of connectivity classes implies that $\gamma_x(A) \in \mathcal{C}$, for all $A \in \mathcal{L}$ and $x \in \mathcal{S}$. Moreover, it is obvious that $C \in \mathcal{C} \Rightarrow \gamma_x(C) = C$, for $x \leq C$. It easily follows from these observations that

$$\mathcal{C} = \bigcup_{x \in \mathcal{S}} \mathrm{Inv}(\gamma_x) = \bigcup_{x \in \mathcal{S}} \{\gamma_x(A) \mid A \in \mathcal{L}\}. \tag{4.9}$$

Below, we show that the connectivity opening $\gamma_x(A)$ corresponds to the connected component of A "marked" by x. First, we define connected components in the context of complete lattices.

4.1.5 Definition. Let $\mathcal{L}$ be a lattice, furnished with a connectivity class $\mathcal{C}$, and let $A \in \mathcal{L}$. A *connected component*, or *grain*, of A is a nonzero element $C \in \mathcal{C}$ such that: (a) $C \leq A$, and (b) there is no $C' \in \mathcal{C}$ such that $C \leq C' \leq A$. △

If C is a grain of A, we write $C \lessdot A$. We have the following result.

4.1.6 Proposition. Let $\mathcal{L}$ be a lattice with sup-generating family $\mathcal{S}$, furnished with a connectivity class $\mathcal{C}$. Given $A \in \mathcal{L}$ and a sup-generator $x \leq A$, $\gamma_x(A)$ is the connected component of A marked by x; i.e., $x \leq \gamma_x(A) \lessdot A$. □

Proof. From the definition of connectivity opening, we have that $x \leq A \Rightarrow x \leq \gamma_x(A)$. In addition, $x \leq A \Rightarrow x \leq \gamma_x(A) \Rightarrow \gamma_x(A) \neq O$ and, as argued previously, $\gamma_x(A) \in \mathcal{C}$. Now, note that $\gamma_x(A) \leq A$, since γ_x is anti-extensive, so that condition (a) in Definition 4.1.5 is satisfied. To show condition (b), let $C' \in \mathcal{C}$ be such that $\gamma_x(A) \leq C' \leq A$. From the definition of connectivity openings, it is clear that $x \leq A$ implies that $x \leq \gamma_x(A)$, which implies $x \leq C' \Rightarrow C' \leq \gamma_x(A) \Rightarrow C' = \gamma_x(A)$. Hence, $x \leq \gamma_x(A) \lessdot A$, as required. Q.E.D.

On the other hand, for any connected component $C \lessdot A$, we have that $C = \gamma_x(A)$, for a sup-generator $x \leq C$. Hence, the connectivity openings completely characterize the connected components of A. Fig. 4.2 illustrates the connectivity opening in the case of lattice $\mathcal{P}(\mathbb{R}^2)$, with the points as sup-generators, where the Euclidean topological connectivity is assumed.

One of the main applications of connectivity is partitioning an object into its connected components. Below, we show that connectivity openings can be used to generate such

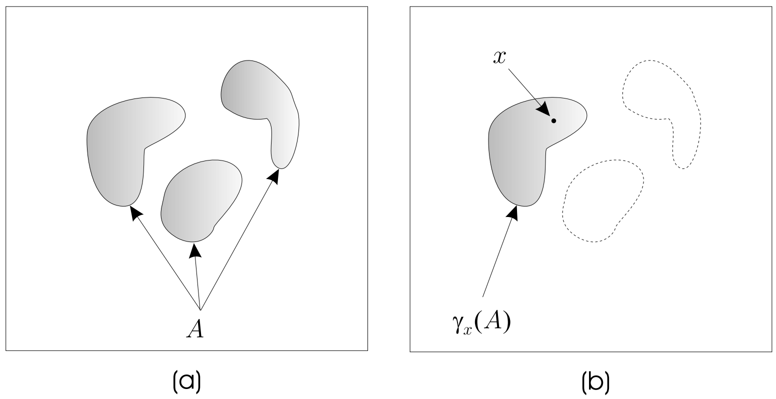


Figure 4.2: (a) A subset A of the 2-D Euclidean space with three connected components. (b) Set $\gamma_x(A)$ is the connected component of A marked by x. The Euclidean topological connectivity is assumed.

partitions. First, however, we need to formally define the concept of a partition. Recall that $\mathcal{S}(A) = \mathcal{S} \cap \mathcal{M}_*(A) = \{x \in \mathcal{S} \mid x \leq A\}$.

4.1.7 Definition. Let $\mathcal{L}$ be a lattice with sup-generating family $\mathcal{S}$. A *partition* of an element $A \in \mathcal{L}$ is a mapping p_A: $\mathcal{S}(A) \to \mathcal{L}$, such that:

(*i*) $x \leq p_A(x) \leq A$, for every $x \in \mathcal{S}(A)$,

(*ii*) $p_A(x) = p_A(y)$ or $p_A(x) \bigwedge p_A(y) = O$, for every $x, y \in \mathcal{S}(A)$.

Each $p_A(x)$ is called a *zone* of the partition p_A of A. We say that p_A is *connected* (with respect to a given connectivity class $\mathcal{C}$) if all zones are connected; i.e., if $p_A(x) \in \mathcal{C}$, for every $x \in \mathcal{S}(A)$. △

Note that

$$A = \bigvee_{x \in \mathcal{S}(A)} p_A(x), \tag{4.10}$$

as a direct consequence of item (*i*) in Definition 4.1.7. Note also that, if $\mathcal{L} = \mathcal{P}(E)$, with the points as sup-generators, then a partition, in the sense of Definition 4.1.7, corresponds to the usual notion of set partition.

For any two partitions p_A and p'_A, we say that p'_A is *finer* than p_A if $p'_A(x) \leq p_A(x)$, for each $x \in \mathcal{S}(A)$; i.e., each zone of p'_A is contained in one of the zones of p_A (we also say that p_A is *coarser* than p'_A). See Fig. 4.3 for an illustration.

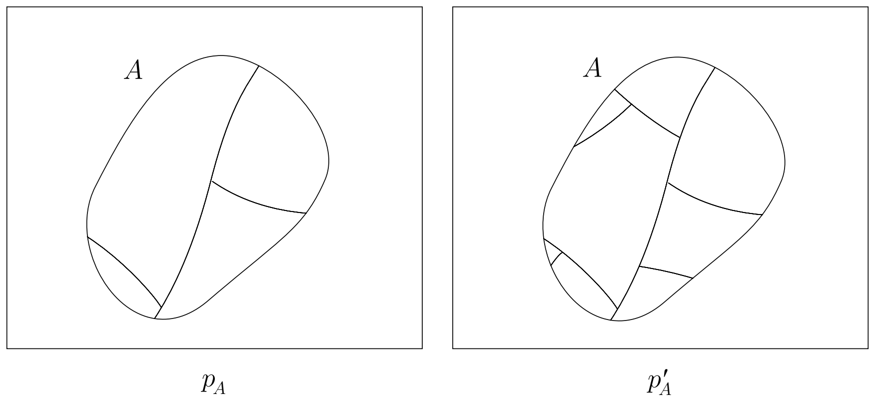


Figure 4.3: The partition p'_A is finer than p_A, since each zone of p'_A is contained in one of the zones of p_A.

It is easy to see that the relationship "finer than" defines a partial order relation $\sqsubseteq$ on the family $\mathcal{P}_A$ of all partitions of A. As a matter of fact, $\mathcal{P}_A$ is a complete lattice: the infimum $\sqcap$ is simply $(\sqcap p^i_A)(x) = \bigwedge p^i_A(x)$, whereas the supremum $\sqcup$ is given by $\sqcup p^i_A = \sqcap\{p_A \mid p_A \sqsupseteq p^i_A,\ i \in I\}$. In other words, the zones of the infimum are obtained by the infima of the zones of the individual partitions, whereas the supremum is the smallest partition that is coarser than each of the individual partitions.

In the following, we show that the connected components of an element $A \in \mathcal{L}$ provide a connected partition of A, which is the coarsest possible connected partition; i.e., it gives the fewest possible number of connected zones.

4.1.8 Proposition. Let $\mathcal{L}$ be a lattice with sup-generating family $\mathcal{S}$, furnished with a connectivity class $\mathcal{C}$, and let $\{\gamma_x \mid x \in \mathcal{S}\}$ be the family of connectivity openings associated with $\mathcal{C}$. For an element $A \in \mathcal{L}$, the mapping c_A: $\mathcal{S}(A) \to \mathcal{L}$, given by

$$c_A(x) = \gamma_x(A), \quad x \in \mathcal{S}(A), \tag{4.11}$$

defines a connected partition of A. Moreover,

$$c_A = \bigsqcup \{c'_A \mid c'_A \text{ is a connected partition of } A\}; \tag{4.12}$$

that is, c_A provides the coarsest possible connected partition of A. □

Proof. That $c_A(x)$ is a connected partition of A follows directly from the definition and properties of the connectivity opening γ_x. To show that c_A is the coarsest possible connected

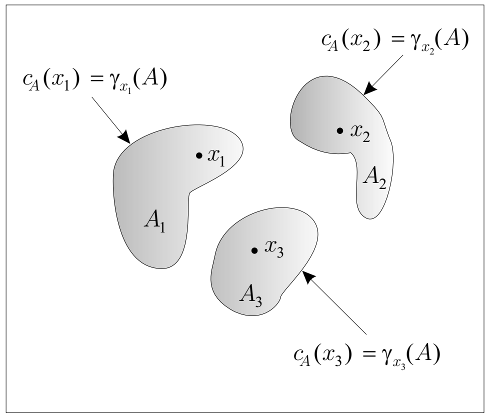


Figure 4.4: A subset A of the 2-D Euclidean space with three connected components A_1, A_2, A_3. The PCC is given by $c_A(x_i) = \gamma_{x_i}(A) = A_i$, for $i = 1, 2, 3$. The Euclidean topological connectivity is assumed.

partition of A, let c'_A be any connected partition of A. For $x \in \mathcal{S}(A)$, we have that $x \leq c_A(x)$ and $x \leq c'_A(x)$, hence $c_A(x) \bigwedge c'_A(x) \geq x \neq O$, which implies that $c_A(x) \bigvee c'_A(x) \in \mathcal{C}$, by axiom (iii) of connectivity classes. However, from the definition of connectivity openings, $x \leq c_A(x) \bigvee c'_A(x) \in \mathcal{C} \Rightarrow c_A(x) \bigvee c'_A(x) \leq \gamma_x(A) = c_A(x) \Rightarrow c'_A(x) \leq c_A(x)$. Since this argument holds for all $x \in \mathcal{S}(A)$, we have shown that c_A is coarser than c'_A. Q.E.D.

The partition c_A is referred to as the *partition of connected components* (PCC) of A. In the binary case (e.g., when $\mathcal{L} = \mathcal{P}(E)$), it will be convenient to consider the PCC c_A as a function from A into $\mathcal{L}$; i.e., $c_A(x) = \gamma_x(A)$, for $x \in A$.

A straightforward corollary to Proposition 4.1.8 is that any element A of a lattice can be recovered from its PCC, since

$$A = \bigvee_{x \in \mathcal{S}(A)} c_A(x) = \bigvee_{x \in \mathcal{S}(A)} \gamma_x(A). \tag{4.13}$$

We have mentioned before that, in the binary case, we fall back into the usual notion of set partition. This is illustrated in Fig. 4.4, where $E = \mathbb{R}^2$ and $\mathcal{C}$ is the Euclidean topological connectivity. The three zones of the PCC correspond to $c_A(x_i) = \gamma_{x_i}(A) = A_i$, for $i = 1, 2, 3$.

As mentioned at the beginning of this section, connectivity classes and connectivity openings are intimately related: connectivity openings characterize, in a unique fashion, the connectivity classes with which they are associated. This is established by the following theorem (recall that $\mathrm{Ccl}(\mathcal{L})$ denotes the set of all connectivity classes in $\mathcal{L}$).

4.1.9 Theorem. Let $\mathcal{L}$ be a lattice with sup-generating family $\mathcal{S}$. For a connectivity class $\mathcal{C} \in \mathrm{Ccl}(\mathcal{L})$, let $\{\gamma_x \mid x \in \mathcal{S}\}$ be the connectivity openings associated with $\mathcal{C}$, given by (4.8). Then,

(*i*) $\gamma_x(x) = x$, for every $x \in \mathcal{S}$,

(*ii*) $x \not\leq A \Rightarrow \gamma_x(A) = O$,

(*iii*) $\gamma_x(A) \bigwedge \gamma_y(A) \neq O \Rightarrow \gamma_x(A) = \gamma_y(A)$ (i.e., $\gamma_x(A)$ and $\gamma_y(A)$ are either equal or disjoint).

Conversely, let $\mathrm{Cop}(\mathcal{L})$ denote the set of all families of openings $\{\gamma_x \mid x \in \mathcal{S}\}$ that satisfy properties (*i*)–(*iii*) above. For $\{\gamma_x \mid x \in \mathcal{S}\} \in \mathrm{Cop}(\mathcal{L})$, let $\mathcal{C}$ be given by (4.9). Then, $\mathcal{C}$ is a connectivity class in $\mathcal{L}$; i.e., $\mathcal{C} \in \mathrm{Ccl}(\mathcal{L})$. Moreover, its family of connectivity openings coincides with $\{\gamma_x \mid x \in \mathcal{S}\}$. Hence, (4.8) and (4.9) establish a bijection between $\mathrm{Ccl}(\mathcal{L})$ and $\mathrm{Cop}(\mathcal{L})$. □

Proof. Clearly, properties (*i*) and (*ii*) are a direct consequence of (4.8). To show property (*iii*), recall that $\gamma_x(A), \gamma_y(A) \in \mathcal{C}$ and $\gamma_x(A), \gamma_y(A) \leq A$. Therefore, by axiom (*iii*) of connectivity classes, we have that $\gamma_x(A) \bigwedge \gamma_y(A) \neq O \Rightarrow C = \gamma_x(A) \bigvee \gamma_y(A) \in \mathcal{C}$, with $C \leq A$. However, since $\gamma_x(A) \bigwedge \gamma_y(A) \neq O$, we have $\gamma_x(A) \neq O$, which implies that $x \leq A$. But $x \leq A \Rightarrow x = \gamma_x(x) \leq \gamma_x(A) \leq C \Rightarrow C \leq \gamma_x(A) \Rightarrow \gamma_y(A) \leq \gamma_x(A)$. The reverse inequality $\gamma_x(A) \leq \gamma_y(A)$ can be shown analogously; hence, $\gamma_x(A) = \gamma_y(A)$.

Now, assume that $\{\gamma_x \mid x \in \mathcal{S}\} \in \mathrm{Cop}(\mathcal{L})$. We show that $\mathcal{C}$, given by (4.9), satisfies axioms (*i*)–(*iii*) of a connectivity class. Clearly, axiom (*i*) follows from the fact that $O \in \mathrm{Inv}(\gamma_x)$. Axiom (*ii*) follows from property (*i*): $\gamma_x(x) = x \Rightarrow x \in \mathrm{Inv}(\gamma_x) \Rightarrow x \in \mathcal{C}$, for all $x \in \mathcal{S}$. To show axiom (*iii*), consider a family $\{C_\alpha\}$ of elements in $\mathcal{C}$ such that $\bigwedge C_\alpha \neq O$. From (4.9), we have that $C_\alpha \in \mathrm{Inv}(\gamma_{x_\alpha}) \Rightarrow C_\alpha = \gamma_{x_\alpha}(C_\alpha)$, for some $x_\alpha \in \mathcal{S}$. Now, since $\bigwedge C_\alpha \neq O$, we can pick $x \in \mathcal{S}$, with $x \leq \bigwedge C_\alpha \leq C_\alpha$. On one hand, this implies that $x \leq \gamma_{x_\alpha}(C_\alpha)$, and on the other hand that $x = \gamma_x(x) \leq \gamma_x(C_\alpha)$. Therefore, $\gamma_x(C_\alpha) \bigwedge \gamma_{x_\alpha}(C_\alpha) \geq x \neq O$. From property (*iii*), it follows that $\gamma_x(C_\alpha) = \gamma_{x_\alpha}(C_\alpha) = C_\alpha \Rightarrow C_\alpha \in \mathrm{Inv}(\gamma_x)$, for all α; therefore, $\bigvee C_\alpha \in \mathrm{Inv}(\gamma_x) \Rightarrow \bigvee C_\alpha \in \mathcal{C}$. Hence, $\mathcal{C} \in \mathrm{Ccl}(\mathcal{L})$. Finally, we show that $\gamma'_x(A) = \bigvee\{C \in \mathcal{C} \mid x \leq C \leq A\} = \gamma_x(A)$, for all $A \in \mathcal{L}$. If $x \not\leq A$, then clearly $\gamma'_x(A) = \gamma_x(A) = O$. So, let $x \leq A$ and define $\mathcal{C}(x, A) = \{C \in \mathcal{C} \mid x \leq C \leq A\}$. Note that $\gamma'_x(A) = \bigvee \mathcal{C}(x, A)$. If $C \in \mathcal{C}(x, A)$, we have $C \in \mathrm{Inv}(\gamma_y)$, for some $y \in \mathcal{S}$, so that $\gamma_y(C) = C \geq x$. On the other hand, $C \geq x \Rightarrow \gamma_x(C) \geq \gamma_x(x) = x$.

Hence, $\gamma_x(C) \bigwedge \gamma_y(C) \neq O$, so that, from property *(iii)*, we have $\gamma_x(C) = \gamma_y(C) = C$. But $C \leq A$, thus $C = \gamma_x(C) \leq \gamma_x(A)$, for all $C \in \mathcal{C}(x, A)$, so that $\gamma'_x(A) \leq \gamma_x(A)$. To show the reverse inequality, note that $x \leq \gamma_x(A) \leq A$, and that $\gamma_x(A) \in \mathrm{Inv}(\gamma_x) \Rightarrow \gamma_x(A) \in \mathcal{C}$. Hence, $\gamma_x(A) \in \mathcal{C}(x, A) \Rightarrow \gamma_x(A) \leq \gamma'_x(A)$. Q.E.D.

In a slightly different form, Theorem 4.1.9 was proved for the binary case in [77, Thm. 2.8], and for the general lattice case in [79, Thm. 3]. This fundamental result shows that connectivity on a lattice $\mathcal{L}$ can be *equivalently* specified by either a connectivity class $\mathcal{C} \in \mathrm{Ccl}(\mathcal{L})$, or by a family of connectivity openings $\{\gamma_x \mid x \in \mathcal{S}\} \in \mathrm{Cop}(\mathcal{L})$.

We remark here that, since $\mathrm{Ccl}(\mathcal{L})$ is a lattice under the inclusion partial order relation (see Proposition 4.1.4), the bijection between $\mathrm{Ccl}(\mathcal{L})$ and $\mathrm{Cop}(\mathcal{L})$ induces a partial order on the set $\mathrm{Cop}(\mathcal{L})$, under which $\mathrm{Cop}(\mathcal{L})$ is also a lattice. These two lattices are of course isomorphic.

We conclude this subsection with a study of semi-continuity properties of connectivity openings. This material will be useful later, especially in Section 4.2.4 dealing with flat grayscale connectivity classes.

First, we introduce the notion of compatible connectivity classes in $\mathcal{P}(E)$.

4.1.10 Definition. Let E be a topological space. A connectivity class $\mathcal{C}$ in $\mathcal{P}(E)$ is said to be *compatible with the topology on* E, if

$$A \in \mathcal{C} \;\Rightarrow\; \overline{A} \in \mathcal{C}. \tag{4.14}$$

△

If no confusion is possible, we simply say that $\mathcal{C}$ is compatible. From Proposition 3.1.3(b), it follows that the topological connectivity class in $\mathcal{P}(E)$ (i.e., the collection of all subsets of E that are topologically connected, see Example 4.1.3(c)) is compatible. The following result gives some useful properties of compatible connectivity classes. Recall that $\mathcal{F}(E)$ is the lattice of closed subsets of E (see Example 2.1.2(b)).

4.1.11 Proposition. Let E be a Hausdorff space, and let $\widehat{\mathcal{C}}$ be a compatible connectivity class in $\mathcal{P}(E)$, with the points as sup-generators. Let $\{\widehat{\gamma}_x \mid x \in E\}$ be the connectivity openings associated with $\widehat{\mathcal{C}}$.

(a) The family $\mathcal{C} = \widehat{\mathcal{C}} \cap \mathcal{F}(E)$ is a connectivity class in $\mathcal{F}(E)$.

(b) The restriction of the connectivity opening $\widehat{\gamma}_x$ to $\mathcal{F}(E)$ defines an operator γ_x on $\mathcal{F}(E)$, for $x \in E$; i.e., $A \in \mathcal{F}(E) \Rightarrow \widehat{\gamma}_x(A) \in \mathcal{F}(E)$, for $x \in E$.

(c) The connectivity openings associated with $\mathcal{C}$ are given by the operators $\{\gamma_x \mid x \in E\}$; i.e., $\gamma_x(A) = \widehat{\gamma}_x(A)$, for $A \in \mathcal{F}(E), x \in E$. □

Proof. (a): Clearly, the empty set is in $\mathcal{C}$. In addition, the points in E are in $\widehat{\mathcal{C}}$ and are closed, since E is Hausdorff; hence, they are also in $\mathcal{C}$. Consider now a family $\{C_\alpha\}$ of elements in $\mathcal{C}$ such that $\bigwedge C_\alpha = \bigcap C_\alpha \neq \emptyset$. Note that $\{C_\alpha\}$ is in $\widehat{\mathcal{C}}$, so that axiom (*iii*) of connectivity classes implies that $\bigcup C_\alpha \in \widehat{\mathcal{C}}$. From (4.14), it follows that $\bigvee C_\alpha = \overline{\bigcup C_\alpha} \in \widehat{\mathcal{C}}$. Since $\bigvee C_\alpha \in \mathcal{F}(E)$, we have that $\bigvee C_\alpha \in \mathcal{C}$, as required.

(b): Let $A \in \mathcal{F}(E)$. If $A = \emptyset$ or $x \notin A$, the result is obvious, so let $x \in A$. From (4.14), we have that $\overline{\widehat{\gamma}_x(A)} \in \widehat{\mathcal{C}}$, since $\widehat{\gamma}_x(A) \in \widehat{\mathcal{C}}$. But $x \in \widehat{\gamma}_x(A) \subseteq \overline{\widehat{\gamma}_x(A)} \Rightarrow x \in \overline{\widehat{\gamma}_x(A)}$, so that $\overline{\widehat{\gamma}_x(A)} \subseteq \widehat{\gamma}_x(A)$, from the definition of connectivity openings (4.8). Hence, $\widehat{\gamma}_x(A) = \overline{\widehat{\gamma}_x(A)}$ and, therefore, $\widehat{\gamma}_x(A) \in \mathcal{F}(E)$.

(c): We show that $\gamma_x(A) = \widehat{\gamma}_x(A)$, for $A \in \mathcal{F}(E)$ and $x \in E$. If $A = \emptyset$ or $x \notin A$, the result is obvious. Thus, let $x \in A$. From part (b), we have that $\widehat{\gamma}_x(A) \in \mathcal{F}(E) \Rightarrow \widehat{\gamma}_x(A) \in \mathcal{C}$, since $\widehat{\gamma}_x(A) \in \widehat{\mathcal{C}}$. From the definition of connectivity openings (4.8) and since $x \in \widehat{\gamma}_x(A) \subseteq A$, we have that $\widehat{\gamma}_x(A) \subseteq \gamma_x(A)$. But $\mathcal{C} \subseteq \widehat{\mathcal{C}}$, so that $\gamma_x(A) \subseteq \widehat{\gamma}_x(A)$ and, therefore, $\gamma_x(A) = \widehat{\gamma}_x(A)$. Q.E.D.

The connectivity class $\mathcal{C}$ is the *restriction* of the connectivity class $\widehat{\mathcal{C}}$ to $\mathcal{F}(E)$. As an example, if $\widehat{\mathcal{C}}$ is the topological connectivity class in $\mathcal{P}(E)$, which is compatible, then $\mathcal{C}$ is the collection of all topologically connected closed subsets of E.

Recall now the definition of lattice upper semi-continuous (l.u.s.c.) operators in Section 2.2.

4.1.12 Proposition. Let $\{\gamma_x \mid x \in \mathcal{S}\}$ be the connectivity openings associated with a strong connectivity class $\mathcal{C}$ in $\mathcal{L}$. For each $x \in \mathcal{S}$, γ_x is l.u.s.c. on $\mathcal{L}$ if and only if $\mathcal{C}$ is such that

$$\bigwedge \mathcal{Q} \in \mathcal{C}, \quad \text{for every totally ordered subset } \mathcal{Q} \subseteq \mathcal{C}. \tag{4.15}$$

□

Proof. "⇒": Let $\mathcal{Q}$ be a totally ordered subset of $\mathcal{C}$. If $\mathcal{Q} = \emptyset$, then $\bigwedge \mathcal{Q} = I \in \mathcal{C}$, since $\mathcal{C}$ is strong. If $\bigwedge \mathcal{Q} = O$, then $\bigwedge \mathcal{Q} \in \mathcal{C}$. Otherwise, pick a sup-generator $x \leq \bigwedge \mathcal{Q} \Rightarrow x \leq A$, for all $A \in \mathcal{Q}$. We have $x \leq A \Rightarrow \gamma_x(A) = A$, for all $A \in \mathcal{Q}$. Since γ_x is l.u.s.c., we have $\bigwedge \mathcal{Q} = \bigwedge_{A \in \mathcal{Q}} A = \bigwedge_{A \in \mathcal{Q}} \gamma_x(A) = \gamma_x(\bigwedge_{A \in \mathcal{Q}} A) = \gamma_x(\bigwedge \mathcal{Q})$, so that $\bigwedge \mathcal{Q} \in \mathcal{C}$, which shows (4.15).

"$\Leftarrow$": Given $x \in \mathcal{S}$, we need to show that $\gamma_x(\bigwedge \mathcal{K}) = \bigwedge_{A \in \mathcal{K}} \gamma_x(A)$, for every totally ordered subset $\mathcal{K}$ of $\mathcal{L}$. If $\mathcal{K} = \emptyset$, the result follows directly from the fact that $\gamma_x(I) = I$, since $\mathcal{C}$ is a strong connectivity class. So let $\mathcal{K}$ be nonempty. If $x \not\leq \bigwedge \mathcal{K}$, then $x \not\leq A$, for some $A \in \mathcal{K}$, so that $\gamma_x(A) = O \Rightarrow \gamma_x(\bigwedge \mathcal{K}) = \bigwedge_{A \in \mathcal{K}} \gamma_x(A) = O$, which shows the desired result. Hence, consider the case when $x \leq \bigwedge \mathcal{K} \Rightarrow x \leq A$, for all $A \in \mathcal{K}$. For every $C \in \mathcal{C}$ such that $x \leq C \leq \bigwedge \mathcal{K}$, we have that $x \leq C \leq A \Rightarrow C \leq \gamma_x(A)$, for all $A \in \mathcal{K}$, which implies $C \leq \bigwedge_{A \in \mathcal{K}} \gamma_x(A)$. Hence, $\gamma_x(\bigwedge \mathcal{K}) = \bigvee\{C \in \mathcal{C} \mid x \leq C \leq \bigwedge \mathcal{K}\} \leq \bigwedge_{A \in \mathcal{K}} \gamma_x(A)$. On the other hand, since γ_x is increasing, $\{\gamma_x(A) \mid A \in \mathcal{K}\}$ is a totally ordered family in $\mathcal{C}$, which implies $\bigwedge_{A \in \mathcal{K}} \gamma_x(A) \in \mathcal{C}$, by virtue of (4.15). In addition, we have $x \leq \gamma_x(A)$, for all $A \in \mathcal{K}$, so that $x \leq \bigwedge_{A \in \mathcal{K}} \gamma_x(A)$. Since $\gamma_x(A) \leq A \Rightarrow \bigwedge_{A \in \mathcal{K}} \gamma_x(A) \leq \bigwedge \mathcal{K}$ and γ_x is increasing, we conclude that $\gamma_x(\bigwedge_{A \in \mathcal{K}} \gamma_x(A)) = \bigwedge_{A \in \mathcal{K}} \gamma_x(A) \leq \gamma_x(\bigwedge \mathcal{K})$. Hence, $\gamma_x(\bigwedge \mathcal{K}) = \bigwedge_{A \in \mathcal{K}} \gamma_x(A)$, as required. Q.E.D.

A similar result for the connectivity openings to be lattice lower semi-continuous (l.l.s.c.) is not possible. In fact, the dual of condition (4.15), namely, $\bigvee \mathcal{Q} \in \mathcal{C}$, for every totally ordered subset $\mathcal{Q} \subseteq \mathcal{C}$, can be shown to hold trivially. However, connectivity openings are not necessarily l.l.s.c., as the following example shows. Let $\mathcal{L} = \mathcal{F}(\mathbb{R}^2)$ be the lattice of closed subsets of $\mathbb{R}^2$ with the Euclidean topology and consider the connectivity class $\mathcal{C}$ of topologically connected closed subsets of $\mathbb{R}^2$ with connectivity openings $\{\gamma_x \mid x \in E\}$ on $\mathcal{F}(\mathbb{R}^2)$. Let $(A_n = [0,1] \times [1/n, 1])_{n \geq 1}$ be a sequence of sets in $\mathcal{L}$, and let $x = (0,0)$. We have that $\gamma_x(\bigvee_{n \geq 1} A_n) = \gamma_x([0,1] \times [0,1]) = [0,1] \times [0,1]$, but $\bigvee_{n \geq 1} \gamma_x(A_n) = \bigvee_{n \geq 1} \emptyset = \emptyset$, so that γ_x is not l.l.s.c.

We can now show that the connectivity openings associated with the connectivity class of topologically connected closed subsets in lattice $\mathcal{F}(E)$ are l.u.s.c., provided that we make suitable assumptions about the topological space E.

4.1.13 Proposition. Let E be a connected compact Hausdorff space, and let $\mathcal{C}$ be the connectivity class of topologically connected closed subsets in lattice $\mathcal{F}(E)$. The connectivity openings $\{\gamma_x \mid x \in E\}$ associated with $\mathcal{C}$ are l.u.s.c. on $\mathcal{F}(E)$; i.e.,

$$\gamma_x(\bigcap A_\alpha) = \bigcap \gamma_x(A_\alpha), \quad x \in E, \tag{4.16}$$

for every family $\{A_\alpha\}$ of totally ordered subsets in $\mathcal{F}(E)$. □

Proof. Since E is connected, then $\mathcal{C}$ is a strong connectivity class. By Proposition 4.1.12, it suffices to show that $\bigwedge \mathcal{Q} \in \mathcal{C}$, for every totally ordered subset $\mathcal{Q} \subseteq \mathcal{C}$. If $\mathcal{Q} = \emptyset$, we have that $\bigwedge \mathcal{Q} = E \in \mathcal{C}$. Moreover, if an $A \in \mathcal{Q}$ is empty, the result follows from the fact that $\bigwedge \mathcal{Q} = \emptyset \in \mathcal{C}$. Hence, we may assume that every $A \in \mathcal{Q}$ is nonempty, in which case $\bigwedge \mathcal{Q} \neq \emptyset$ by Proposition 2.3.7(a). Suppose that $L = \bigwedge \mathcal{Q}$ is disconnected. Then, there is a separation of L into open sets G and H in E. Let $C = L \setminus H = L \cap G$ and $D = L \setminus G = L \cap H$. Since L is closed in E, C and D are nonempty disjoint closed sets in E. From Proposition 2.3.6, there are two disjoint open sets U, V in E such that $C \subset U$ and $D \subset V$. Now, we have that $L \subset U \cup V$. Since $U \cup V$ is an open set in E, we can use Proposition 2.3.7(b) to conclude that there is some $A \in \mathcal{Q}$ such that $A \subset U \cup V$. But, we have that $L \subseteq A$, which implies $A \cap U \neq \emptyset$ and $A \cap V \neq \emptyset$. Therefore, U, V provide a separation of A, which is a contradiction. Hence, $\bigwedge \mathcal{Q}$ must be connected, and since it is also closed, we have that $\bigwedge \mathcal{Q} \in \mathcal{C}$, as required. Q.E.D.

Note that Proposition 4.1.11(b) implies that, for a compatible connectivity class $\mathcal{C}$ in $\mathcal{P}(E)$ and for $A \in \mathcal{F}(E)$, the PCC $c_A(x) = \gamma_x(A)$ defines a function from A into $\mathcal{F}(E)$. Therefore, its continuity properties can be assessed by using the H-M topology on $\mathcal{F}(E)$ (see Section 2.4). In particular, we have the following result, which will be useful later.

4.1.14 Proposition. Let E be a compact Hausdorff space with a countable basis, and let $\mathcal{C}$ be a compatible connectivity class in $\mathcal{P}(E)$, with connectivity openings $\{\gamma_x \mid x \in E\}$, such that the PCC $c_A(x) = \gamma_x(A)$ is an H-M u.s.c. function from A into $\mathcal{F}(E)$, for every $A \in \mathcal{F}(E)$. Let B be an arbitrary subset of E. Any open set in E that contains a connected component of $\overline{B}$ contains a connected component of B as well. □

Proof. Let $C = \gamma_x(\overline{B})$ be a connected component of $\overline{B}$, where $x \in \overline{B}$. Since E has a countable basis, it follows from Proposition 2.3.3(a) that there is a sequence (x_n) of points in B such that $x_n \to x$. Let U be an open set that contains C. Note that $C = c_{\overline{B}}(x)$ misses the closed set U^c. Since $c_{\overline{B}}$ is an H-M u.s.c. function from $\overline{B}$ into $\mathcal{F}(E)$, we can apply Proposition 2.4.5(a) to conclude that $c_{\overline{B}}(x_n)$ misses U^c, eventually. This implies that there exists some $x_{n_0} \in B$ such that $c_{\overline{B}}(x_{n_0}) = \gamma_{x_{n_0}}(\overline{B}) \subset U$. Note that $\gamma_{x_{n_0}}(B) \subseteq \gamma_{x_{n_0}}(\overline{B})$, since γ_x is increasing and $B \subseteq \overline{B}$. But $x_{n_0} \in B$, so $\gamma_{x_{n_0}}(B)$ is a connected component of B contained in U. Q.E.D.

As we mentioned previously, topological connectivity is compatible. In addition, it satisfies the requirements of the previous proposition. This is shown by the following result.

4.1.15 Proposition. Let E be a connected compact Hausdorff space with a countable basis, and $\mathcal{C}$ be the topological connectivity class in $\mathcal{P}(E)$, with connectivity openings $\{\gamma_x \mid x \in E\}$. The PCC $c_A(x) = \gamma_x(A)$ is an H-M u.s.c. function from A into $\mathcal{F}(E)$, for every $A \in \mathcal{F}(E)$. □

Proof. Let $A \in \mathcal{F}(E)$ and $(x_i)_{i \in \mathbb{Z}_+}$ be a sequence of points in A such that $x_i \to x \in A$. We need to show that $\overline{\lim}\, c_A(x_i) \subseteq c_A(x)$; i.e., $\overline{\lim}\, \gamma_{x_i}(A) \subseteq \gamma_x(A)$. We show that $L = \overline{\lim}\, \gamma_{x_i}(A)$ is topologically connected. First, note that $x \in L$. Suppose that L is not topologically connected. Then, there is a separation of L into open sets G and H. Let $C = L \setminus H = L \cap G$ and $D = L \setminus G = L \cap H$. Take $x \in C$ and pick a $y \in D$. Since L is closed, C and D are nonempty disjoint closed sets. From Proposition 2.3.6, there are two disjoint open sets U and V such that $x \in C \subset U$ and $y \in D \subset V$, with $L \subset U \cup V$. Since $y \in \overline{\lim}\, \gamma_{x_i}(A)$, there is a subsequence $(y_{i_k} \in \gamma_{x_{i_k}}(A))$ converging to y, with $x_{i_k} \to x$. Let i_1 (resp. i_2) be an index such that $x_{i_k} \in U$, for all $i_k \geq i_1$ (resp. $y_{i_k} \in V$, for all $i_k \geq i_2$), and let $i_0 = \max\{i_1, i_2\}$. Note that, for $i_k \geq i_0$, we have $\gamma_{x_{i_k}}(A) \cap U \neq \emptyset$ and $\gamma_{x_{i_k}}(A) \cap V \neq \emptyset$. Since $\gamma_{x_{i_k}}(A)$ is topologically connected, we must have $\gamma_{x_{i_k}}(A) \setminus (U \cup V) \neq \emptyset$, for all $i_k \geq i_0$; otherwise, U, V would be a separation of $\gamma_{x_{i_k}}(A)$. Hence, we can pick a sequence $(z_{i_k} \in \gamma_{x_{i_k}}(A) \setminus (U \cup V))_{i_k \geq i_0}$, which must have an accumulation point z in E, since E is compact. Note that z must be necessarily outside of $U \cup V$, since $z_{i_k} \in (U \cup V)^c$, which is a closed set. But $z \in L = \overline{\lim}\, \gamma_{x_i}(A)$, which implies that $L \not\subseteq U \cup V$, a contradiction. Therefore, $\overline{\lim}\, \gamma_{x_i}(A)$ must be topologically connected. Recall that $x \in L = \overline{\lim}\, \gamma_{x_i}(A)$. Since $\overline{\lim}\, \gamma_{x_i}(A)$ is topologically connected, we have that $\overline{\lim}\, \gamma_{x_i}(A) \subseteq \gamma_x(A)$. Q.E.D.

Note that the PCC function $c_A(x) = \gamma_x(A)$ associated with a topological connectivity class need not be H-M l.s.c. For instance, let E be any connected, closed and bounded subspace of $\mathbb{R}^2$, furnished with the Euclidean topological connectivity, large enough to contain the set $A = (\{0\} \times [0, 4/3]) \cup (K \times [0, 1])$, where $K = \{1/i \mid i \in \mathbb{Z}_+\}$. See Fig. 4.5 for an illustration. Note that $A \in \mathcal{F}(E)$. Consider the sequence $(x_i = (1/i, 0))_{i \in \mathbb{Z}_+}$. Clearly, $\gamma_{x_i}(A) = \{1/i\} \times [0, 1]$, with $\underline{\lim}\, \gamma_{x_i}(A) = \{0\} \times [0, 1]$. But $x_i \to x = (0, 0)$, and $\gamma_x(A) = \{0\} \times [0, 4/3]$, so that $\gamma_x(A) \not\subseteq \underline{\lim}\, \gamma_{x_i}(A)$.

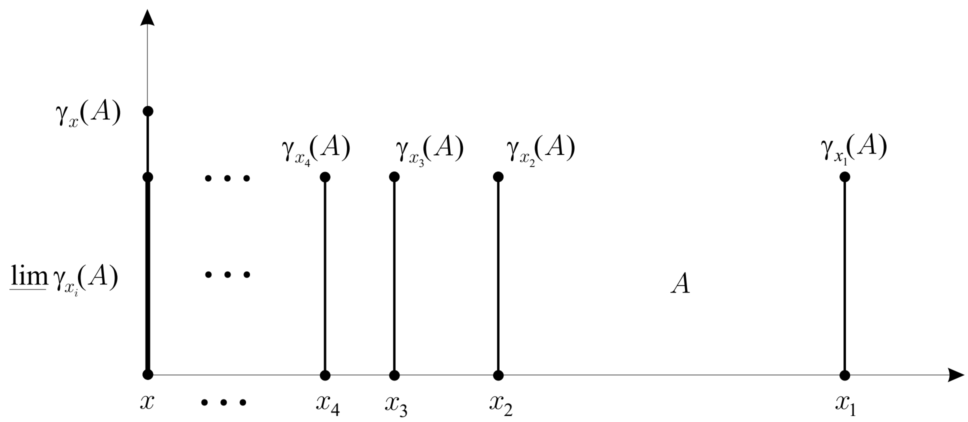


Figure 4.5: An example which shows that the PCC function $c_A(x) = \gamma_x(A)$ associated with topological connectivity need not be H-M l.s.c. In this example, $A = (\{0\} \times [0, 4/3]) \cup (K \times [0, 1])$, where $K = \{1/i \mid i \in \mathbb{Z}_+\}$.

4.1.3 Reconstruction

Connectivity openings give rise to the important notion of *reconstruction*, one of the most useful morphological tools for image analysis applications [34, 35, 74, 89]. As we have seen, connectivity openings extract connected components marked by sup-generators. The basic idea behind reconstruction is to extend this principle to include arbitrary markers, not just sup-generators. Given an element $M \in \mathcal{L}$, called a *marker*, the *reconstruction* $\rho(A \mid M)$ of an element $A \in \mathcal{L}$ from M is given by

$$\rho(A \mid M) \;=\; \bigvee_{x \leq M} \gamma_x(A). \tag{4.17}$$

Note that $\rho(A \mid O) = \bigvee \emptyset = O$. Moreover, it is easy to see that $\rho(A \mid M) = \rho(A \mid M \wedge A)$, so that $\rho(A \mid M) = O$, if $M \wedge A = O$. As a consequence of this property, it is usually assumed that $M \leq A$. Note that, as a consequence of properties (*ii*) and (*iii*) of Theorem 4.1.9, we have that

$$\gamma_x(A) = \begin{cases} \rho(A \mid x), & \text{if } \; x \leq A \\ O, & \text{otherwise} \end{cases}. \tag{4.18}$$

In particular, when the marker M is a sup-generator x, the reconstruction operator $\rho(A \mid x)$ reduces to the connectivity opening $\gamma_x(A)$, provided that $x \leq A$. Note that, if $M = x \not\leq A$, then it is not in general true that $\rho(A \mid x) = \gamma_x(A)$ (consider for instance the grayscale support connectivity of Example 4.1.3(h)). However, it can be easily verified that, for atomic lattices, such as $\mathcal{L} = \mathcal{P}(E)$, we have that $\rho(A \mid x) = \gamma_x(A)$, for all $x \in \mathcal{S}$.

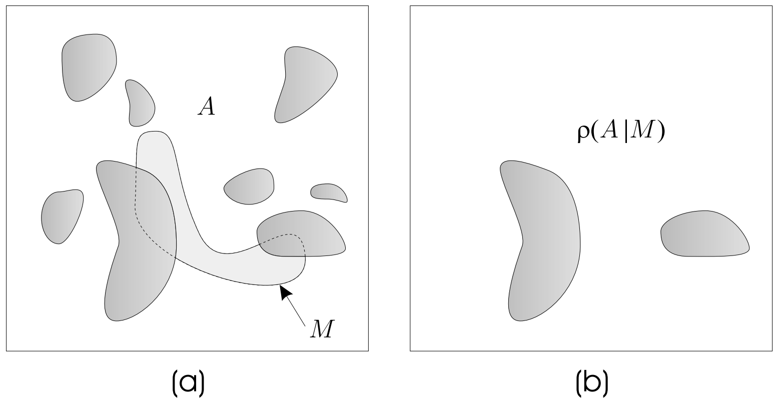


Figure 4.6: (a) A subset A of the 2-D Euclidean space and a marker M. (b) The reconstruction $\rho(A \mid M)$ of A from M. The Euclidean topological connectivity is assumed.

The following result shows that the reconstruction operator $\rho(A \mid M)$ extracts the connected components of A that "intersect" marker M.

4.1.16 Proposition. If ρ is the reconstruction operator, defined by (4.17), then

$$\rho(A \mid M) = \bigvee \{C \lessdot A \mid C \wedge M \neq O\}. \tag{4.19}$$

□

Proof. If $C \lessdot A$ and $C \wedge M \neq O$, we can pick a sup-generator $x \leq C \wedge M \leq C$, so that $C = \gamma_x(A)$, with $x \leq C \wedge M \Rightarrow x \leq M$. This shows that $\rho(A \mid M) = \bigvee_{x \leq M} \gamma_x(A) \geq \bigvee \{C \lessdot A \mid C \wedge M \neq O\}$. To show the reverse inequality, recall that $\rho(A \mid M) = \rho(A \mid M \wedge A) = \bigvee_{x \leq M \wedge A} \gamma_x(A)$. We have that $x \leq M \wedge A \Rightarrow x \leq A \Rightarrow x \leq \gamma_x(A)$. But $\gamma_x(A) \lessdot A$, with $\gamma_x(A) \wedge M \geq x \neq O$. Hence, $\rho(A \mid M) \leq \bigvee \{C \lessdot A \mid C \wedge M \neq O\}$. Q.E.D.

The reconstruction operator is illustrated in Fig. 4.6. Note that (4.19) is satisfied.

Since reconstruction generalizes, to some extent, connectivity openings, which carry all information about the associated connectivity class (see Theorem 4.1.9), it is not surprising that reconstruction may carry this information as well. In fact, this is true in infinite $\vee$-distributive lattices, as it is shown by the following theorem (recall that $\mathrm{Cop}(\mathcal{L})$ denotes the set of all families of connectivity openings on $\mathcal{L}$).

4.1.17 Theorem. Let $\mathcal{L}$ be an infinite $\vee$-distributive lattice with sup-generating family $\mathcal{S}$. Let $\rho(A \mid M)$ be the reconstruction operator, given by (4.17), associated with a family of connectivity openings $\{\gamma_x \mid x \in \mathcal{S}\} \in \mathrm{Cop}(\mathcal{L})$. Then, (*i*) $M \wedge A \leq \rho(A \mid M)$,

(*ii*) $\rho(\cdot \mid M)$ is an opening,

(*iii*) $\rho(A \mid \cdot)$ is increasing and idempotent,

(*iv*) $\rho(A \mid \cdot)$ is symmetric; i.e., $y \leq \rho(A \mid x) \;\Leftrightarrow\; x \leq \rho(A \mid y)$, for $x, y \leq A$,

(*v*) $\rho(A \mid M) \;=\; \bigvee_{x \leq M} \rho(A \mid x)$.

Conversely, let $\mathrm{Rec}(\mathcal{L})$ denote the set of all operators ρ: $\mathcal{L} \times \mathcal{L} \to \mathcal{L}$ that satisfy properties (*i*)–(*v*) above. For $\rho \in \mathrm{Rec}(\mathcal{L})$, let $\{\gamma_x \mid x \in \mathcal{S}\}$ be the family of operators given by (4.18). Then, $\{\gamma_x \mid x \in \mathcal{S}\}$ is a family of connectivity openings on $\mathcal{L}$; i.e., $\{\gamma_x \mid x \in \mathcal{S}\} \in \mathrm{Cop}(\mathcal{L})$. Moreover, its reconstruction operator coincides with ρ. Hence, (4.17) and (4.18) establish a bijection between $\mathrm{Cop}(\mathcal{L})$ and $\mathrm{Rec}(\mathcal{L})$. □

Proof. To show property (*i*), note that $x \leq M \wedge A \Rightarrow x \leq A \Rightarrow x \leq \gamma_x(A)$. Hence, $M \wedge A = \bigvee_{x \leq M \wedge A} x \leq \bigvee_{x \leq M \wedge A} \gamma_x(A) = \rho(A \mid M \wedge A) = \rho(A \mid M)$.

Property (*ii*) follows from the fact that the supremum of openings is an opening [34].

To show property (*iii*), note that $M_1 \leq M_2 \Rightarrow \rho(A \mid M_1) = \bigvee_{x \leq M_1} \gamma_x(A) \leq \bigvee_{x \leq M_2} \gamma_x(A) = \rho(A \mid M_2)$, so that $\rho(A \mid \cdot)$ is increasing. To show idempotence, note that, from (*i*) and the fact that $\rho(A \mid \cdot)$ is increasing, we have that $A \wedge M \leq \rho(A \mid M) \Rightarrow \rho(A \mid M) = \rho(A \mid A \wedge M) \leq \rho(A \mid \rho(A \mid M))$. To establish the reverse inequality, we will show that, for any $C \lessdot A$, $C \wedge \rho(A \mid M) \neq O \Rightarrow C \wedge M \neq O$, which gives the desired result, since, from (4.19), it would follow that $\rho(A \mid \rho(A \mid M)) = \bigvee\{C \lessdot A \mid C \wedge \rho(A \mid M) \neq O\} \leq \bigvee\{C \mid C \lessdot A, C \wedge M \neq O\} = \rho(A \mid M)$. We will show the contrapositive $C \wedge M = O \Rightarrow C \wedge \rho(A \mid M) = O$, for $C \lessdot A$. We have that $C \wedge \rho(A \mid M) = C \wedge \bigvee\{C_\alpha \mid C_\alpha \lessdot A, C_\alpha \wedge M \neq O\} = \bigvee\{C \wedge C_\alpha \mid C_\alpha \lessdot A, C_\alpha \wedge M \neq O\}$, by the infinite $\vee$-distributivity of $\mathcal{L}$. Now, since $C \wedge M = O$, we have $C \neq C_\alpha$, for all α, and since C and C_α are all grains of A, it follows that $C \wedge C_\alpha = O$, for all α, so that $C \wedge \rho(A \mid M) = O$, as required.

To show property (*iv*), notice that, since $x, y \leq A$, it follows from (4.18) that $\rho(A \mid x) = \gamma_x(A)$ and $\rho(A \mid y) = \gamma_y(A)$. Hence, we need to show that $y \leq \gamma_x(A) \;\Leftrightarrow\; x \leq \gamma_y(A)$. We have that $y \leq \gamma_x(A) \leq A \Rightarrow y \leq \gamma_y(A) \Rightarrow \gamma_y(A) \wedge \gamma_x(A) \geq y \neq O \Rightarrow \gamma_y(A) = \gamma_x(A)$. In addition, $y \leq \gamma_x(A) \Rightarrow \gamma_x(A) \neq O$, so that $x \leq \gamma_x(A) = \gamma_y(A)$. The proof of the reverse implication is similar.

To show property (v), notice that $x \le M \Rightarrow \rho(A \mid x) \le \rho(A \mid M) \Rightarrow \bigvee_{x \le M} \rho(A \mid x) \le \rho(A \mid M)$, since $\rho(A \mid \cdot)$ is increasing. The reverse inequality follows from $\rho(A \mid M) = \rho(A \mid M \wedge A) = \bigvee_{x \le M \wedge A} \gamma_x(A) = \bigvee_{x \le M \wedge A} \rho(A \mid x) \le \bigvee_{x \le M} \rho(A \mid x)$, where we used (4.18).

To show the second part of the result, assume that $\rho \in \mathrm{Rec}(\mathcal{L})$. We show that $\{\gamma_x \mid x \in \mathcal{S}\}$, given by (4.18), are openings that satisfy the properties listed in Theorem 4.1.9, namely: (a) $\gamma_x(x) = x$, (b) $x \not\le A \Rightarrow \gamma_x(A) = O$, and (c) $\gamma_x(A) \wedge \gamma_y(A) \ne O \Rightarrow \gamma_x(A) = \gamma_y(A)$. By property (ii) of ρ, we have that $\rho(\cdot \mid x)$ is an opening, which implies that γ_x is also an opening, for $x \in \mathcal{S}$. To show (a), note that $\gamma_x(x) = \rho(x \mid x)$. But property (i) of ρ gives $x \le \rho(x \mid x)$, whereas, from property (ii), we get $\rho(x \mid x) \le x$, so that $\rho(x \mid x) = x$. Condition (b) is satisfied by definition. To show (c), note that $\gamma_x(A) \wedge \gamma_y(A) \ne O$ implies $\gamma_x(A) \ne O$ and $\gamma_y(A) \ne O$, so that $\gamma_x(A) = \rho(A \mid x)$ and $\gamma_y(A) = \rho(A \mid y)$ (and $x \le A$, $y \le A$). Hence, we need to show that $\rho(A \mid x) \wedge \rho(A \mid y) \ne O \Rightarrow \rho(A \mid x) = \rho(A \mid y)$. Let $z \le \rho(A \mid x) \wedge \rho(A \mid y)$. From property (iii) of ρ, we have that $z \le \rho(A \mid x) \Rightarrow \rho(A \mid z) \le \rho(A \mid \rho(A \mid x)) = \rho(A \mid x)$. On the other hand, recall that $x \le A$, and $z \le \rho(A \mid x) \le A$, so we can apply property (iv) of ρ to get $z \le \rho(A \mid x) \Rightarrow x \le \rho(A \mid z)$, so that $\rho(A \mid x) \le \rho(A \mid \rho(A \mid z)) = \rho(A \mid z)$, as before. Hence, $\rho(A \mid x) = \rho(A \mid z)$. By using the same chain of reasoning, we get that $z \le \rho(A \mid y) \Rightarrow \rho(A \mid y) = \rho(A \mid z)$. Therefore, $\rho(A \mid x) = \rho(A \mid y)$. Hence, $\{\gamma_x \mid x \in \mathcal{S}\} \in \mathrm{Cop}(\mathcal{L})$. Finally, we show that $\rho'(A \mid M) = \bigvee_{x \le M} \gamma_x(A) = \rho(A \mid M)$, for all $A, M \in \mathcal{L}$. We have that $\rho'(A \mid M) = \rho'(A \mid M \wedge A) = \bigvee_{x \le M \wedge A} \gamma_x(A) = \bigvee_{x \le M \wedge A} \rho(A \mid x) = \rho(A \mid M \wedge A) = \rho(A \mid M)$, where we used (4.18) and property (v) of ρ. Q.E.D.

In a slightly different form, Theorem 4.1.17 was established for the binary case in [35, Prop. 5.1]. Note that the infinite $\vee$-distributivity of $\mathcal{L}$ is needed only to establish the idempotence of $\rho(A \mid \cdot)$; i.e., to show that (4.17) gives a mapping from $\mathrm{Cop}(\mathcal{L})$ into $\mathrm{Rec}(\mathcal{L})$. In particular, (4.18) maps $\mathrm{Rec}(\mathcal{L})$ into $\mathrm{Cop}(\mathcal{L})$, without the assumption of infinite $\vee$-distributivity of $\mathcal{L}$.

We remark that a similar result was independently obtained by C. Ronse and J. Serra in [64, Thm. 19]. The basic differences between our result and theirs are: (1) Ronse and Serra do not assume infinite $\vee$-distributivity for $\mathcal{L}$; (2) they define a set $\mathrm{Rec}^*(\mathcal{L})$ of reconstruction operators on $\mathcal{L}$ that includes, in a different form, properties (i)–(iv) of Theorem 4.1.17, but leaves out property (v), so that $\mathrm{Rec}(\mathcal{L}) \subseteq \mathrm{Rec}^*(\mathcal{L})$. As a result, Theorem 19 in [64] does not establish a bijection from $\mathrm{Rec}^*(\mathcal{L})$ to $\mathrm{Cop}(\mathcal{L})$, but only a surjection (in the case when $\mathcal{L}$ is infinite $\vee$-distributive). In other words, two different reconstruction operators, in the sense of [64], can give rise to the same family of connectivity openings; i.e., to the same connectivity on $\mathcal{L}$, a situation that is ruled out by property (v) of Theorem 4.1.17. As a matter of fact, property (v) is satisfied whenever $\rho(A \mid \cdot)$ is a dilation on $\mathcal{L}$ (in the case of strongly semi-atomic lattices, such as $\mathcal{L} = \mathcal{P}(E)$ or $\mathcal{L} = \mathrm{Fun}(E, \overline{\mathbb{Z}})$, we can show that property (v) is equivalent to $\rho(A \mid \cdot)$ being a dilation; e.g., see H. Heijmans' original binary result in [35, Prop. 5.1]). Note that, since the inclusion $\mathrm{Rec}(\mathcal{L}) \subseteq \mathrm{Rec}^*(\mathcal{L})$ can be strict, more examples of reconstruction operators are allowed by the framework in [64] (in particular, that framework allows examples where $\rho(A \mid \cdot)$ is not a dilation).

Theorems 4.1.9 and 4.1.17 show that connectivity on an infinite $\vee$-distributive lattice $\mathcal{L}$ can be *equivalently* specified by: (1) a connectivity class $\mathcal{C} \in \mathrm{Ccl}(\mathcal{L})$; (2) a family of connectivity openings $\{\gamma_x \mid x \in \mathcal{S}\} \in \mathrm{Cop}(\mathcal{L})$; or (3) a reconstruction operator $\rho \in \mathrm{Rec}(\mathcal{L})$. Given any one of these three equivalent ways of specifying a connectivity, one can move liberally among all three.

We remark here that, since $\mathrm{Cop}(\mathcal{L})$ is a lattice under the partial order induced by the inclusion partial order relation in $\mathrm{Ccl}(\mathcal{L})$ (see remarks after Theorem 4.1.9), the bijection between $\mathrm{Cop}(\mathcal{L})$ and $\mathrm{Rec}(\mathcal{L})$ induces a partial order relation on the set $\mathrm{Rec}(\mathcal{L})$, under which $\mathrm{Rec}(\mathcal{L})$ is also a lattice. Of course, these three lattices are isomorphic to each other.

Below, we give two examples of reconstruction operators that lead to connectivities.

4.1.18 Example.

(a) Let $\mathcal{L} = \mathcal{P}(\mathbb{Z}^n)$, with the points as sup-generators. Consider a symmetric adjacency relation on $\mathbb{Z}^n$, such as the adjacency relation given by the edges of an undirected graph $G = (\mathbb{Z}^n, L)$ – see Section 3.2. Let the operator ρ: $\mathcal{P}(\mathbb{Z}^n) \times \mathcal{P}(\mathbb{Z}^n) \to \mathcal{P}(\mathbb{Z}^n)$ be defined by the following *propagation algorithm* (see also [35]). Given $A, M \subseteq \mathbb{Z}^n$:

```
R = M ∩ A;
repeat
   N = {all points in A ∖ R that are adjacent to R};
   R = R ∪ N;
until N = ∅;
ρ(A | M) = R;
```

It is easy to see that $\rho \in \mathrm{Rec}(\mathcal{L})$. The associated connectivity corresponds to graph-theoretic connectivity on $\mathbb{Z}^n$, associated with a graph $G = (\mathbb{Z}^n, L)$, where L is given by the symmetric adjacency relation under consideration. In practice, this is the most common form of binary reconstruction encountered in digital image analysis, where the adjacency relation is given, for example, by the 4- or 8-adjacency relations on $\mathbb{Z}^2$, discussed in Section 3.2.

(b) Let $\mathcal{L}$ be an infinite $\vee$-distributive lattice, and δ be an extensive dilation on $\mathcal{L}$. For a given $A \in \mathcal{L}$, the operator $\delta_A(M) = \delta(M) \wedge A$ is clearly a dilation on $\mathcal{L}$, known as the *geodesic dilation* of the marker M inside the *mask* A. Define the operator ρ: $\mathcal{L} \times \mathcal{L} \to \mathcal{L}$ by

$$\rho(A \mid M) = \bigvee_{i=1}^{\infty} \delta_A^i(M), \tag{4.20}$$

where ψ^i denotes the composition of operator ψ i times. The operator $\rho(A \mid M)$ is known as the *geodesic reconstruction* of the marker M inside the mask A. It is clear that $\rho(A \mid \cdot)$ is a dilation on $\mathcal{L}$, since it is a supremum of dilations [34]. It follows that property (v) of Theorem 4.1.17 is satisfied. Under certain (not too demanding) additional conditions on the dilation δ, given in [64], one can show that properties (i)–(iv) are satisfied as well. In other words, $\rho \in \mathrm{Rec}(\mathcal{L})$, so that it gives rise to a connectivity class in $\mathcal{L}$. ◊

Example 4.1.18(b) is taken from [64]. It generalizes (to the case of infinite $\vee$-distributive lattices) the well-known concept of binary geodesic reconstruction [34] that has been extensively used in image analysis as a tool for extracting connected components in binary images. As a matter of fact, Example 4.1.18(a) is a special case of this general framework, where the dilation δ on $\mathcal{P}(\mathbb{Z}^n)$ is given by $\delta(M) = M \cup \{v \in \mathbb{Z}^n \mid v \text{ is adjacent to a point } w \in M\}$, for $M \subseteq \mathbb{Z}^n$.

We conclude this section by discussing the well-known grayscale reconstruction operator, which is very useful in applications of image processing and analysis [89]. Given a connectivity class in $\mathcal{P}(E)$, and the associated reconstruction operator ρ: $\mathcal{P}(E) \times \mathcal{P}(E) \to \mathcal{P}(E)$, we define the operator $\widetilde{\rho}$: $\mathrm{Fun}(E, \mathcal{T}) \times \mathrm{Fun}(E, \mathcal{T}) \to \mathrm{Fun}(E, \mathcal{T})$ by

$$\widetilde{\rho}(f \mid g)(v) = \bigvee \{t \in \mathcal{T} \mid v \in \rho(X_t(f) \mid X_t(g))\}, \quad v \in E, \tag{4.21}$$

where X_t: $\mathrm{Fun}(E, \mathcal{T}) \to \mathcal{P}(E)$ is the threshold operator, given by $X_t(f) = \{v \in E \mid f(v) \geq t\}$, for $f \in \mathrm{Fun}(E, \mathcal{T})$. If we assume that $\mathcal{T}$ is a chain, then the operator $\widetilde{\rho}(f \mid g)$

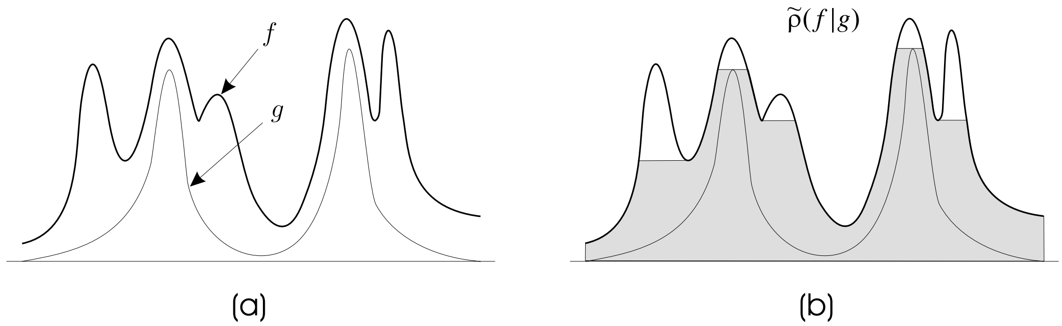


Figure 4.7: (a) Original image f and a marker g. (b) The grayscale reconstruction $\widetilde{\rho}\,(f \mid g)$, according to the usual topological connectivity of the Euclidean real line.

in (4.21) is known as the *grayscale reconstruction* of f from marker g. As can be easily seen, $\widetilde{\rho}\,(\cdot \mid g)$ is the semi-flat operator generated by the family of binary operators $\{\psi_t(\cdot) = \rho(\cdot \mid X_t(g)) \mid t \in \mathcal{T}\}$ (see Section 2.2). From Proposition 2.2.13(a) and item (*ii*) of Theorem 4.1.17, it follows that $\widetilde{\rho}\,(\cdot \mid g)$ is an opening on $\mathrm{Fun}(E, \mathcal{T})$. Fig. 4.7 illustrates the grayscale reconstruction operator in the one-dimensional case. The connectivity assumed here is the usual topological connectivity of the Euclidean real line. Although there is no counterpart to (4.19) for grayscale reconstruction, we could state, in loose terms, that grayscale reconstruction recovers the "connected peaks" of f, according to the assumed connectivity, "marked by" g, as can be seen in Fig. 4.7.

4.2 Connectivity Classes in ψ-Invariant Lattices

Given a lattice $\mathcal{L}$, finding a family $\mathcal{C} \subseteq \mathcal{L}$ such that axiom (*iii*) of connectivity classes is satisfied depends on the supremum and infimum operations associated with $\mathcal{L}$. We now present a method that allows us to construct a new lattice $\mathcal{L}^{\psi}$, from a given lattice $\mathcal{L}$, where the infimum operation in $\mathcal{L}$ is suitably "modified" so that axiom (*iii*) becomes less restrictive. The new lattice $\mathcal{L}^{\psi}$, to be referred to as a ψ-invariant lattice, is constructed by means of an appropriately chosen operator ψ and allows us to develop interesting examples of connectivity classes, including a novel example of connectivity for grayscale images. (in [79, 80], J. Serra also introduced examples of grayscale connectivity classes in function lattices other than $\mathrm{Fun}(E, \mathcal{T})$, using an approach different from ours).

4.2.1 ψ-Invariant Lattices

Recall from Section 2.2 the definition of the characteristic opening ψ° associated with an operator ψ. We have the following result.

4.2.1 Proposition. Given an operator ψ on a lattice $\mathcal{L}$ such that $\text{Inv}(\psi)$ is sup-closed, $\mathcal{L}^\psi = \text{Inv}(\psi)$ is an underlattice of $\mathcal{L}$, with supremum $\bigvee^\psi$ and infimum $\bigwedge^\psi$, given by

$$\bigvee\nolimits^\psi A_\alpha = \bigvee A_\alpha \quad \text{and} \quad \bigwedge\nolimits^\psi A_\alpha = \psi^\circ\left(\bigwedge A_\alpha\right), \tag{4.22}$$

where ψ° is the characteristic opening associated with ψ, given by (2.15). □

Proof. Since $\text{Inv}(\psi)$ is sup-closed, $\mathcal{L}^\psi$ is non-empty, and therefore $\mathcal{L}^\psi$ is a poset with the partial order relation of $\mathcal{L}$. Moreover, if $\{A_\alpha\} \subseteq \mathcal{L}^\psi$, we have that $\bigvee A_\alpha \in \mathcal{L}^\psi$, so that $\bigvee^\psi A_\alpha = \bigvee A_\alpha$. For the infimum in $\mathcal{L}^\psi$, we have that

$$\begin{aligned} \bigwedge\nolimits^\psi A_\alpha &= \bigvee\nolimits^\psi \{B \in \mathcal{L}^\psi \mid B \leq A_\alpha, \forall\alpha\} \\ &= \bigvee \{B \in \text{Inv}(\psi) \mid B \leq \bigwedge A_\alpha\} \\ &= \psi^\circ\left(\bigwedge A_\alpha\right), \end{aligned} \tag{4.23}$$

which shows the desired result. Q.E.D.

Lattice $\mathcal{L}^\psi$ is called the *ψ-invariant lattice* associated with ψ. From Proposition 2.2.5, we have that $\text{Inv}(\psi^\circ) = \langle \text{Inv}(\psi) \mid \vee \rangle = \text{Inv}(\psi) = \mathcal{L}^\psi$. Hence, $\bigwedge^\psi A_\alpha = \bigwedge A_\alpha$ if and only if $\bigwedge A_\alpha \in \mathcal{L}^\psi$. Otherwise, from the anti-extensivity of ψ°, the infimum in $\mathcal{L}^\psi$ is *strictly less* than the infimum in $\mathcal{L}$.

The characteristic opening ψ° acts as a *projection* from the base lattice $\mathcal{L}$ onto the new lattice $\mathcal{L}^\psi$. Given an $A \in \mathcal{L}$, it is easy to see from (2.15) that $\psi^\circ(A)$ is the largest element of $\mathcal{L}^\psi$ that is smaller than A. This property will be of importance later, when we construct connectivity classes in ψ-invariant lattices. This construction requires preprocessing by ψ° in order to bring a given element of $\mathcal{L}$ into $\mathcal{L}^\psi$.

Notice that Proposition 4.2.1 requires that ψ be such that $\text{Inv}(\psi)$ is sup-closed. From Proposition 2.2.4, the increasing and anti-extensivity properties of ψ are sufficient conditions for $\text{Inv}(\psi)$ to be sup-closed. However, they are not necessary. In fact, we later present an example of a grayscale connectivity class in a ψ-invariant lattice for which ψ is not increasing, but for which $\text{Inv}(\psi)$ is sup-closed.

An important special case of a ψ-invariant lattice is obtained when ψ is an opening θ. As we mentioned above, Proposition 2.2.4 guarantees that $\mathrm{Inv}(\theta)$ is sup-closed. By employing Proposition 2.2.6(c), we obtain the following corollary to Proposition 4.2.1 (this result also corresponds to Proposition 2.2.2(b)).

4.2.2 Corollary. If θ is an opening on $\mathcal{L}$, then $\mathcal{L}^\theta = \mathrm{Inv}(\theta)$ is an underlattice of $\mathcal{L}$, with supremum $\bigvee^\theta$ and infimum $\bigwedge^\theta$, given by

$$\bigvee{}^\theta A_\alpha = \bigvee A_\alpha \quad \text{and} \quad \bigwedge{}^\theta A_\alpha = \theta\left(\bigwedge A_\alpha\right). \tag{4.24}$$

□

We conclude this subsection with an important observation. Although the supremum in a ψ-invariant lattice $\mathcal{L}^\psi$ is the same as the supremum in lattice $\mathcal{L}$, the infimum in $\mathcal{L}^\psi$ is less than the infimum in $\mathcal{L}$. Clearly, it is possible that $\bigwedge^\psi A_\alpha = O$ even though $\bigwedge A_\alpha \neq O$, so that the infimum in the new lattice $\mathcal{L}^\psi$ may be thought of as being *less restrictive* with respect to axiom (*iii*) of connectivity classes. In the following subsections, it will become clear that this fact can be exploited in order to construct new examples of connectivity classes.

4.2.2 B-open Connectivity Class

Let us take $\mathcal{L} = \mathcal{P}(E)$, with $E = \mathbb{R}^n$ or $\mathbb{Z}^n$, and consider the θ-invariant lattice $\mathcal{L}^\theta = \mathrm{Inv}(\theta)$, where θ is the structural opening $\theta(A) = A \circ B$. Notice that

$$\mathcal{L}^\theta = \{A \oplus B \mid A \in \mathcal{P}(E)\}. \tag{4.25}$$

From Corollary 4.2.2, $\mathcal{L}^\theta$ is an underlattice of $\mathcal{P}(E)$, where the supremum is set union and the infimum is the structural opening of set intersection with the structuring element B. Lattice $\mathcal{L}^\theta$ is sup-generated by the family $\mathcal{S}^\theta = \{B_v \mid v \in E\}$, where B_v denotes the translation of B to a point $v \in E$. Note that $\mathcal{S}^\theta$ is composed of atoms, so that $\mathcal{L}^\theta$ is atomic.

The following result provides a connectivity class in $\mathcal{L}^\theta$.

4.2.3 Proposition. Let $\mathcal{C}$ be a connectivity class in $\mathcal{P}(E)$ and assume that $B_v \in \mathcal{C}$, for all $v \in E$. Then, $\mathcal{C}^\theta = \mathcal{C} \cap \mathcal{L}^\theta$ is a connectivity class in $\mathcal{L}^\theta$. □

Proof. Clearly, $\emptyset \in \mathcal{C}^\theta$ and $\mathcal{S}^\theta \subseteq \mathcal{C}^\theta$. To show axiom (*iii*) of connectivity classes, consider a family $\{C_\alpha\}$ in $\mathcal{C}^\theta$ such that $\bigwedge^\theta C_\alpha = (\bigcap C_\alpha) \circ B \neq \emptyset$. Since θ is anti-extensive, this

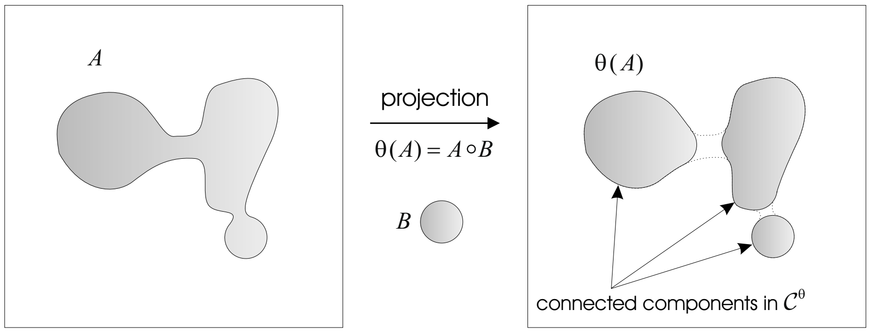


Figure 4.8: An example of B-open connectivity. Note that, although A consists of one connected component in $\mathcal{C}$, A is segmented into three connected components in $\mathcal{C}^\theta$. Moreover, the union of these three connected components does not reconstruct the original object A, but its projection $\theta(A)$. The Euclidean topological connectivity is assumed for $\mathcal{C}$.

implies that $\bigcap C_\alpha \supseteq (\bigcap C_\alpha) \circ B \neq \emptyset$. But since $C_\alpha \in \mathcal{C}$, for all α, we conclude that $\bigvee^\theta C_\alpha = \bigcup C_\alpha \in \mathcal{C}$, so that $\bigvee^\theta C_\alpha \in \mathcal{C}^\theta$. Q.E.D.

The connectivity class $\mathcal{C}^\theta$ consists of all B-open connected elements of $\mathcal{P}(E)$, according to $\mathcal{C}$. It will therefore be referred to as the *B-open connectivity class*. Fig. 4.8 depicts an example of B-open connectivity, with the Euclidean topological connectivity assumed for $\mathcal{C}$. Note that, although A consists of one connected component in $\mathcal{C}$, A is segmented into three connected components in $\mathcal{C}^\theta$. Moreover, the union of these three connected components does not reconstruct the original object A, but its projection $\theta(A)$.

4.2.3 Graph-Theoretic k-Connectivity Class

Recall the concept of graph-theoretic k-connectivity, discussed in Chapter 3. We show here that this classical notion leads to an interesting example of a connectivity class.

Let us take $\mathcal{L} = \mathcal{P}(\mathbb{Z}^n)$, and consider the θ-invariant lattice $\mathcal{L}^\theta = \mathrm{Inv}(\theta)$, where θ is a *discrete area opening*, given by

$$\theta(A) = \begin{cases} A, & \text{if } |A| \geq k \\ \emptyset, & \text{otherwise} \end{cases}, \tag{4.26}$$

with $|A|$ being the number of points in A and k being a positive integer. Notice that

$$\mathcal{L}^\theta = \{A \in \mathcal{P}(\mathbb{Z}^n) \mid A = \emptyset \text{ or } |A| \geq k\}. \tag{4.27}$$

Therefore, $\mathcal{L}^\theta$ contains the empty set and the subsets of $\mathbb{Z}^n$ that contain at least k points. From Corollary 4.2.2, $\mathcal{L}^\theta$ is an underlattice of $\mathcal{P}(\mathbb{Z}^n)$, with the supremum being set union and the infimum given by

$$\bigwedge^{\theta} A_\alpha = \begin{cases} \bigcap A_\alpha, & \text{if } |\bigcap A_\alpha| \geq k \\ \emptyset, & \text{otherwise} \end{cases}. \tag{4.28}$$

Lattice $\mathcal{L}^\theta$ is sup-generated by the family $\mathcal{S}^\theta = \{A \in \mathcal{P}(\mathbb{Z}^n) \mid |A| = k\}$; i.e., the sup-generators are the subsets of $\mathbb{Z}^n$ that have exactly k points. Note that $\mathcal{S}^\theta$ is composed of atoms, so that $\mathcal{L}^\theta$ is atomic.

The following result provides a connectivity class in $\mathcal{L}^\theta$.

4.2.4 Proposition. Consider a graph $G = (\mathbb{Z}^n, L)$. The family

$$\mathcal{C}^\theta = \mathcal{S}^\theta \cup \{A \in \mathcal{L}^\theta \mid A \text{ is } k\text{-connected in } G\} \tag{4.29}$$

is a connectivity class in $\mathcal{L}^\theta$. □

PROOF. By definition of k-connectivity, $\emptyset \in \mathcal{C}^\theta$. In addition, $\mathcal{S}^\theta \subseteq \mathcal{C}^\theta$, by (4.29). This shows axioms (i) and (ii) of connectivity classes. Moreover, axiom (iii) can be shown easily with the help of Proposition 3.2.11(a). Q.E.D.

We refer to $\mathcal{C}^\theta$ as the *graph-theoretic k-connectivity class*. Fig. 4.9 depicts an example of graph-theoretic k-connectivity, for $n = 2$ and $k = 2$, where 8-adjacency connectivity is assumed for G. Note that $A \in \mathcal{L}^\theta$ is not 2-connected. It consists of two 2-connected components C_1 and C_2, with $C_1 \cap C_2 \neq \emptyset$; but $C_1 \bigwedge^\theta C_2 = \emptyset$. Therefore, although the two 2-connected components C_1, C_2 overlap in $\mathcal{L} = \mathcal{P}(\mathbb{Z}^2)$, they do not overlap in $\mathcal{L}^\theta$. Overlapping of connected components may add extra flexibility in certain applications. For example, overlapping between two grains may be due to noise, in which case the two components should be considered disjoint.

4.2.4 Flat Grayscale Connectivity Class

In this subsection, we provide an example of a connectivity class for grayscale functions, which is more meaningful than the grayscale "support" connectivity of Example 4.1.3(h). Essentially, this example is constructed by extending a binary connectivity class to a grayscale one by using threshold decomposition. Hence, the name "flat grayscale connectivity class." It turns out that, in our scheme, the grayscale connected components of an

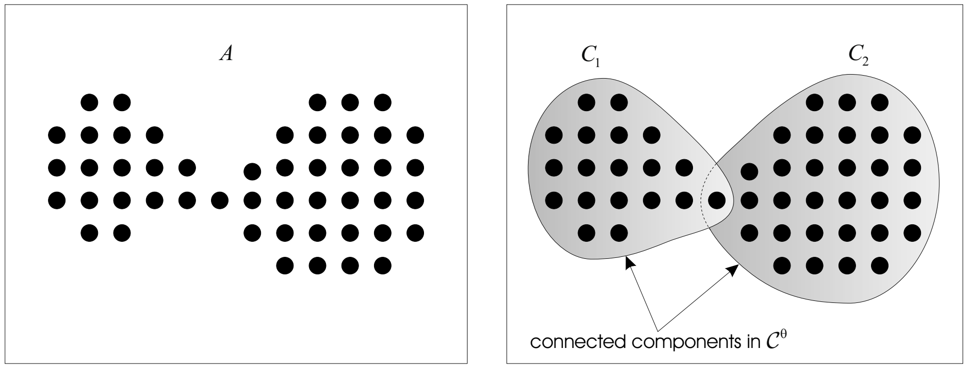


Figure 4.9: An example of graph-theoretic k-connectivity, with $k = 2$. Although $A \in \mathcal{L}^\theta$ is not 2-connected, it consists of two 2-connected components C_1 and C_2, which overlap in $\mathcal{L} = \mathcal{P}(\mathbb{Z}^2)$ but do not overlap in $\mathcal{L}^\theta$.

image will be defined in terms of its regional maxima. We remark that using regional maxima to define grayscale connected components is an established principle in the literature of image processing and analysis. For instance, see [38, 47].

Recall the lattice $\mathrm{Fun}_u(E, \mathcal{T})$ of upper semi-continuous functions, discussed in Example 2.1.2(e). Here, as our base lattice $\mathcal{L}$, we adopt the lattice $\mathrm{Fun}_u(E, \overline{\mathbb{R}})$ of (extended) real-valued upper semi-continuous functions on E. We need to characterize the supremum in $\mathrm{Fun}_u(E, \overline{\mathbb{R}})$. Before that, however, we show the following lemma.

4.2.5 Lemma. Let $f \in \mathrm{Fun}(E, \overline{\mathbb{R}})$ be a function generated by a decreasing family of sets $\{A(t)\}_{t \in \overline{\mathbb{R}}}$. The smallest u.s.c. function $g \in \mathrm{Fun}(E, \overline{\mathbb{R}})$ greater than f is generated by the family of sets $\{\overline{A(t)}\}_{t \in \overline{\mathbb{R}}}$. □

Proof. First, note that, according to Proposition 2.1.3(a), g is indeed u.s.c. since, by Proposition 2.2.11(a), $X_t(g) = \bigcap_{s<t} \overline{A(s)}$, so that $X_t(g)$ is a closed set, for all $t \in \overline{\mathbb{R}}$. Moreover, $X_t(g) = \bigcap_{s<t} \overline{A(s)} \supseteq \bigcap_{s<t} A(s) = X_t(f)$, for all $t \in \overline{\mathbb{R}}$, so that, by Proposition 2.2.12(a), $g \geq f$. Now, let g' be any u.s.c. function such that $g' \geq f$. For all $t \in \overline{\mathbb{R}}$, $X_t(g') \supseteq X_t(f) = \bigcap_{s<t} A(s) \supseteq A(t) \Rightarrow X_t(g') \supseteq \overline{A(t)}$, since $X_t(g')$ is closed. But this implies that $X_t(g') = \bigcap_{s<t} X_s(g') \supseteq \bigcap_{s<t} \overline{A(s)} = X_t(g)$, for all $t \in \overline{\mathbb{R}}$, or $g' \geq g$. Hence, g is the smallest u.s.c. function greater than f. Q.E.D.

We now have the following result.

4.2.6 Proposition. The supremum $\bigvee_u f_\alpha$ in $\text{Fun}_u(E, \overline{\mathbb{R}})$ of a family $\{f_\alpha\}$ in $\text{Fun}_u(E, \overline{\mathbb{R}})$ is the function generated by the sets $\left\{\overline{\bigcup X_t(f_\alpha)}\right\}_{t \in \overline{\mathbb{R}}}$. □

PROOF. Note that, by definition of the supremum, $\bigvee_u f_\alpha = \bigwedge_u \{f \in \text{Fun}_u(E, \overline{\mathbb{R}}) \mid f \geq f_\alpha, \forall \alpha\} = \bigwedge \{f \text{ is u.s.c.} \mid f \geq \bigvee f_\alpha\}$. From Proposition 2.2.12(b), $\bigvee f_\alpha$ is the function in $\text{Fun}(E, \overline{\mathbb{R}})$ generated by the sets $\{\bigcup X_t(f_\alpha)\}_{t \in \overline{\mathbb{R}}}$. It then follows from Lemma 4.2.5 that $\bigvee_u f_\alpha$ is the function in $\text{Fun}_u(E, \overline{\mathbb{R}})$ generated by the sets $\left\{\overline{\bigcup X_t(f_\alpha)}\right\}_{t \in \overline{\mathbb{R}}}$. Q.E.D.

Of course, it is not in general true that $X_t(\bigvee_u f_\alpha) = \overline{\bigcup X_t(f_\alpha)}$, for all $t \in \overline{\mathbb{R}}$. In addition, note that $f_1 \vee_u f_2 = f_1 \vee f_2$, for any $f_1, f_2 \in \text{Fun}_u(E, \overline{\mathbb{R}})$, where $\vee$ denotes the pointwise supremum in $\text{Fun}(E, \overline{\mathbb{R}})$. In general, whenever $\bigvee f_\alpha$ is u.s.c., then $\bigvee_u f_\alpha = \bigvee f_\alpha$. Hence, the supremum $\bigvee_u$ in lattice $\text{Fun}_u(E, \overline{\mathbb{R}})$ can, and often does, reduce to the familiar pointwise supremum $\bigvee$. Finally, recall that the infimum $\bigwedge_u$ in lattice $\text{Fun}_u(E, \overline{\mathbb{R}})$ is always the familiar pointwise infimum $\bigwedge$.

From this point on, we assume that E is a connected compact Hausdorff space with a countable basis (for example, E is a connected, closed and bounded subset of $\mathbb{R}^n$, with the Euclidean topology), and that $\mathcal{C}$ is a compatible connectivity class in $\mathcal{P}(E)$, such that:

(a) The connectivity openings γ_x are $\downarrow$-continuous on $\mathcal{F}(E)$, for all $x \in E$.

(b) The PCC function $c_A(x) = \gamma_x(A)$ is an H-M u.s.c. function from A into $\mathcal{F}(E)$, for all $A \in \mathcal{F}(E)$.

For example, the topological connectivity class in $\mathcal{P}(E)$ satisfies these requirements, as shown in Propositions 4.1.13 and 4.1.15.

Next, we define the notion of a regional maximum.

4.2.7 Definition. The set $R \subseteq E$ is a *regional maximum of* $f \in \text{Fun}_u(E, \overline{\mathbb{R}})$ *at level* $t \in \overline{\mathbb{R}}$ if R is a connected component of $X_t(f)$, according to $\mathcal{C}$, and $R \cap X_s(f) = \emptyset$, for all $s > t$. △

Clearly, the concept of a regional maximum depends on the underlying connectivity class $\mathcal{C}$. Note that R is a closed set, since $X_t(f)$ is closed, for all $t \in \overline{\mathbb{R}}$, and $\mathcal{C}$ is compatible (see Propositions 2.1.3(a) and 4.1.11(b)). It is easy to see that, if R is a regional maximum of $f \in \text{Fun}_u(E, \overline{\mathbb{R}})$, then f is constant over R. We denote this constant value by $f(R)$. In addition, we denote by $\mathcal{R}(f)$ the set of all regional maxima of a function f, according to $\mathcal{C}$, and by $\mathcal{R}_t(f)$ the set of all regional maxima of f that are above level t; i.e., $\mathcal{R}_t(f) = \{R \in \mathcal{R}(f) \mid f(R) \geq t\}$, for $t \in \overline{\mathbb{R}}$. We now have the following result.

4.2.8 Proposition. Let $f \in \mathrm{Fun}_u(E, \overline{\mathbb{R}})$.

(a) The function f has at least one regional maximum.

(b) The function f has exactly one regional maximum if and only if $X_t(f) \in \mathcal{C}$, for all $t \in \mathcal{T}$. □

PROOF. (a): From Weierstrass' theorem of real analysis [39] and the facts that E is compact and f is an u.s.c. function, f achieves its supremum in E; i.e., there is a point $x_0 \in E$ such that $f(x_0) = \bigvee\{f(x) \mid x \in E\}$. It is clear that $X_t(f) = \emptyset$, for all $t > f(x_0)$. Hence, $R = \gamma_{x_0}(X_{f(x_0)}(f))$ is a regional maximum of f at level $f(x_0)$.

(b): We show that the function $f \in \mathrm{Fun}_u(E, \overline{\mathbb{R}})$ has two or more regional maxima if and only if $X_t(f) \notin \mathcal{C}$, for some $t \in \overline{\mathbb{R}}$, which is the contrapositive of the assertion. To show the direct implication, assume that R_1 and R_2 are two regional maxima of f. If $f(R_1) = f(R_2) = t$, then $X_t(f) \notin \mathcal{C}$. Otherwise, let $f(R_1) = t_1 > t_2 = f(R_2)$. We have that $R_1 \subseteq X_{t_1}(f) \subseteq X_{t_2}(f)$. But $R_2 \cap X_{t_1}(f) = \emptyset \Rightarrow R_1 \cap R_2 = \emptyset$, so that R_2 must be a strict subset of $X_{t_2}(f)$, which implies that $X_{t_2}(f) \notin \mathcal{C}$. To show the converse implication, assume that $X_t(f) \notin \mathcal{C}$, for some $t \in \overline{\mathbb{R}}$, and let C_1 and C_2 be two connected components of $X_t(f)$. Sets C_1 and C_2 are closed and therefore compact, according to Proposition 2.3.4(a). Hence, the restrictions f_1 and f_2 of f to C_1 and C_2, respectively, are u.s.c. functions defined on compact sets, so that each achieves its supremum, say at points $x_1 \in R_1$ and $x_2 \in R_2$. Clearly, the corresponding regional maxima of f_1 and f_2 at $f(x_1)$ and $f(x_2)$, respectively, are distinct regional maxima of f. Q.E.D.

For a given parameter $q \in \overline{\mathbb{R}}$, consider the operator

$$\psi(f) = \begin{cases} f, & \text{if } \mathcal{R}(f) = \mathcal{R}_q(f) \\ O, & \text{otherwise} \end{cases}, \quad f \in \mathrm{Fun}_u(E, \overline{\mathbb{R}}). \tag{4.30}$$

In other words, ψ leaves f unchanged if all regional maxima of f are at level q or above, and produces the null function otherwise. Note that ψ is anti-extensive and idempotent, but not increasing; hence, it is not an opening. However, we have the following result.

4.2.9 Proposition. Let ψ be as in (4.30). Then, $\mathrm{Inv}(\psi)$ is sup-closed in $\mathrm{Fun}_u(E, \overline{\mathbb{R}})$. □

PROOF. First, note that $\psi(O) = O$ so that $O \in \mathrm{Inv}(\psi)$. Let $\{f_\alpha\}$ be a family of functions in $\mathrm{Fun}_u(E, \overline{\mathbb{R}})$ such that $\{f_\alpha\} \subseteq \mathrm{Inv}(\psi)$. We can assume, without loss of generality, that all f_α are nonzero. Hence, $\mathcal{R}_q(f_\alpha) = \mathcal{R}(f_\alpha) \neq \emptyset$, for each f_α, which implies that $X_t(f_\alpha) \neq \emptyset$, for all $t \leq q$. Let $f = \bigvee_u f_\alpha$, where $\bigvee_u$ is the supremum in lattice $\mathrm{Fun}_u(E, \overline{\mathbb{R}})$. From Propositions 2.2.11(a) and 4.2.6, we have that $X_t(f) = \bigcap_{s<t} \overline{\bigcup X_s(f_\alpha)} \supseteq \bigcup X_t(f_\alpha)$. Therefore, $X_t(f) \neq \emptyset$, for all $t \leq q$. Suppose that R is a regional maximum of f at a level $r < q$. By definition, we have that $R \cap X_t(f) = \emptyset$, for all $t > r$. Therefore, the sets R and $T = X_q(f)$ are closed nonempty disjoint sets. Moreover, by Proposition 2.3.6, there exist disjoint open sets U and V such that $R \subset U$ and $T \subset V$. Now, given $x \in R$, we have that $R = \gamma_x(X_r(f)) = \gamma_x(\bigcap_{s<r} \overline{\bigcup X_s(f_\alpha)}) = \bigcap_{s<r} \gamma_x(\overline{\bigcup X_s(f_\alpha)})$, from the $\downarrow$-continuity of γ_x on $\mathcal{F}(E)$ and Proposition 2.2.10. Let $C(s) = \gamma_x(\overline{\bigcup X_s(f_\alpha)})$, for $s < r$. Note that $\{C(s)\}_{s<r}$ is a decreasing family of nonempty closed sets in the compact space E, so that we can use Proposition 2.3.7(b) to conclude that there is some $p < r$ such that $C(p) \subset U$. Since the PCC function is H-M u.s.c., we can apply Proposition 4.1.14 to conclude that there is some connected component C of $\bigcup X_p(f_\alpha)$ such that $C \subset U$. Clearly, this implies that there is some index α' such that a connected component C' of $X_p(f_{\alpha'})$ is contained in U. This follows from the fact that each component of $\bigcup A_\alpha$ must contain at least one component of some $A_{\alpha'}$. However, note that $T = X_q(f) \supseteq \bigcup X_q(f_\alpha)$ implies that $X_q(f_\alpha) \subset V$, for all α. Hence, $C' \cap X_q(f_{\alpha'}) = \emptyset$, so that function $f_{\alpha'}$ has a regional maximum inside C' at some level below t, which is a contradiction. Therefore, $f = \bigvee_u f_\alpha$ must not have any regional maxima below level q, in which case $\psi(f) = f$ and, therefore, $f \in \mathrm{Inv}(\psi)$. Q.E.D.

From Proposition 2.2.6(c), since ψ is not an opening, the associated characteristic opening ψ°, given by (2.15), will be different from ψ on $\mathrm{Fun}_u(E, \overline{\mathbb{R}})$. We will investigate the form of ψ° next. Recall that a *cylinder* $h_{A,t}$ of base $A \subseteq E$ and height $t \in \overline{\mathbb{R}}$ is a function from E into $\overline{\mathbb{R}}$ defined by

$$h_{A,t}(x) = \begin{cases} t, & \text{if } x \in A, \\ 0, & \text{otherwise} \end{cases}, \quad \text{for } x \in E. \tag{4.31}$$

We need the following auxiliary result.

4.2.10 Lemma. Let $f \in \mathrm{Fun}_u(E, \overline{\mathbb{R}})$, and let $\widetilde{\rho}$: $\mathrm{Fun}(E, \overline{\mathbb{R}}) \times \mathrm{Fun}(E, \overline{\mathbb{R}}) \to \mathrm{Fun}(E, \overline{\mathbb{R}})$ be the grayscale reconstruction operator, given by (4.21).

(a) The function

$$g = \widetilde{\rho}(f \mid h_{R,f(R)}), \quad R \in \mathcal{R}(f), \tag{4.32}$$

is u.s.c., and $\mathcal{R}(g) = \{R\}$, with $g(R) = f(R)$.

(b) We have that

$$f = \bigvee_u \{\widetilde{\rho}(f \mid h_{R,f(R)}) \mid R \in \mathcal{R}(f)\}. \tag{4.33}$$

□

PROOF. (a): From the definition of $\widetilde{\rho}$ in (4.21), we have that

$$g(v) = \bigvee \{t \in \overline{\mathbb{R}} \mid v \in \rho(\, X_t(f) \mid X_t(h_{R,f(R)}))\}, \quad v \in E. \tag{4.34}$$

Note that $X_t(h_{R,f(R)}) = R$, if $t \leq f(R)$, and $X_t(h_{R,f(R)}) = \emptyset$, if $t > f(R)$. Also, $X_t(f) \cap R = \emptyset$, for $t > f(R)$. Hence, $\rho(X_t(f) \mid X_t(h_{R,f(R)})) = \rho(X_t(f) \mid R)$, for all $t \in \overline{\mathbb{R}}$. Moreover, R is connected, so that it must be contained in one of the grains of $X_t(f)$ and, therefore, $\rho(X_t(f) \mid R) = \gamma_x(X_t(f))$, for some $x \in R$. Thus, (4.34) becomes $g(v) = \bigvee\{t \in \overline{\mathbb{R}} \mid v \in \gamma_x(X_t(f))\}$, for $v \in E$. Hence, $X_t(g) = \bigcap_{s<t} \gamma_x(X_s(f)) = \gamma_x\left(\bigcap_{s<t} X_s(f)\right) = \gamma_x(X_t(f))$, for all $t \in \overline{\mathbb{R}}$, from the $\downarrow$-continuity of γ_x on $\mathcal{F}(E)$ and Proposition 2.2.10. In other words, $X_t(g)$ is a closed (by compatibility of $\mathcal{C}$) connected set, for all $t \in \overline{\mathbb{R}}$, so that, from Propositions 2.1.3(a) and 4.2.8(b), g is u.s.c. and has a single regional maximum. In addition, we have that $X_t(g) = R$, for $t = f(R)$, and $X_t(g) = \emptyset$, for $t > f(R)$, so that R is the only regional maximum of g at level $g(R) = f(R)$.

(b): First, note that the right-hand side of (4.33) makes sense since, from (a), $\widetilde{\rho}(f \mid h_{R,f(R)})$ is a function in $\mathrm{Fun}_u(E, \overline{\mathbb{R}})$, for each $R \in \mathcal{R}(f)$. Let C be a connected component of any nonempty threshold set $X_t(f)$ of f. It follows from Proposition 2.3.4(a), and the compatibility of $\mathcal{C}$, that C is compact. In addition, the restriction of f to C is an u.s.c. function; hence, C contains some regional maximum $R \in \mathcal{R}_t(f)$. Moreover, the definition of regional maximum implies that each $R \in \mathcal{R}_t(f)$ must be contained in some component C of $X_t(f)$. Since $X_t(f)$ equals the union of its components, we conclude that $X_t(f) = \bigcup_{R \in \mathcal{R}_t(f)} \rho(X_t(f) \,|\, R)$. But, by definition, any $R \in \mathcal{R}(f) \setminus \mathcal{R}_t(f)$ does not intersect $X_t(f)$. Hence,

$$X_t(f) = \bigcup_{R \in \mathcal{R}(f)} \rho(X_t(f) \,|\, R). \tag{4.35}$$

Now, from the proof of part (a), we have that

$$X_t(\,\widetilde{\rho}(f \mid h_{R,f(R)})) = \rho(X_t(f) \,|\, R), \quad \text{for all } t \in \overline{\mathbb{R}}. \tag{4.36}$$

It follows from Propositions 2.2.11(a) and 4.2.6, and (4.35), (4.36), that $X_t(\bigvee_u\{\widetilde{\rho}(f \mid h_{R,f(R)}) \mid R \in \mathcal{R}(f)\}\,) = \bigcap_{s<t} \overline{\bigcup_{R \in \mathcal{R}(f)} X_t(\,\widetilde{\rho}(f \mid h_{R,f(R)}))} = \bigcap_{s<t} \overline{\bigcup_{R \in \mathcal{R}(f)} \rho(X_s(f) \,|\, R)} = \bigcap_{s<t} \overline{X_s(f)} = \bigcap_{s<t} X_s(f) = X_t(f)$, for all $t \in \overline{\mathbb{R}}$, which implies (4.33). Q.E.D.

We can now provide an expression for the characteristic opening ψ° associated with the operator ψ given by (4.30).

4.2.11 Proposition. For the operator ψ given by (4.30), the associated characteristic opening ψ° is given by

$$\psi^\circ(f) = \bigvee_u \{\widetilde{\rho}(f \mid h_{R,f(R)}) \mid R \in \mathcal{R}_q(f)\}, \quad f \in \mathrm{Fun}_u(E, \overline{\mathrm{I\!R}}). \tag{4.37}$$

□

Proof. Consider the operator $\theta(f) = \bigvee_u \{\widetilde{\rho}(f \mid h_{R,f(R)}) \mid R \in \mathcal{R}_q(f)\}$. Note that Lemma 4.2.10(a) guarantees that θ is an operator on $\mathrm{Fun}_u(E, \overline{\mathrm{I\!R}})$.

We first show that θ is an increasing operator. Let $f, g \in \mathrm{Fun}_u(E, \overline{\mathrm{I\!R}})$ such that $f \leq g$. Consider a regional maximum $R \in \mathcal{R}_q(f)$ at level $t = f(R)$. Since $R \in \mathcal{C}$ and $R \subseteq X_t(f) \subseteq X_t(g)$, we must have that $R \subseteq C$, for some connected component C of $X_t(g)$. As argued in the proof of Lemma 4.2.10(b), there is a regional maximum $R' \in \mathcal{R}_q(g)$ such that $R' \subseteq C$. For any $s \leq t$, it is clear that $\rho(X_s(g) \mid R) = \rho(X_s(g) \mid R')$, since both R and R' are contained in the same connected component of $X_s(g)$ that contains C. From (4.36), this implies that $X_s(\widetilde{\rho}(f \mid h_{R,f(R)})) = \rho(X_s(f) \mid R) \subseteq \rho(X_s(g) \mid R) = \rho(X_s(g) \mid R') = X_s(\widetilde{\rho}(g \mid h_{R',g(R')}))$, for all $s \leq f(R)$, where we have used the fact that $\rho(\cdot \mid R)$ is an opening (see Theorem 4.1.17) and thus increasing. Since $X_s(\widetilde{\rho}(f \mid h_{R,f(R)})) = \emptyset$, for $s > f(R)$, we conclude that $\widetilde{\rho}(f \mid h_{R,f(R)}) \leq \widetilde{\rho}(g \mid h_{R',g(R')})$. This implies that $\theta(f) \leq \theta(g)$.

Now, let $f \in \mathrm{Fun}_u(E, \overline{\mathrm{I\!R}})$. If $\mathcal{R}_q(f) = \emptyset$, it is easy to verify that (4.37) holds trivially. Assume that $\mathcal{R}_q(f) \neq \emptyset$ and let ψ° be the characteristic opening associated with the operator ψ given by (4.30). From Propositions 2.2.5 and 4.2.9, we have that $\mathrm{Inv}(\psi^\circ) = \mathrm{Inv}(\psi)$, and since $\psi^\circ(f) \in \mathrm{Inv}(\psi^\circ)$, $\psi^\circ(f) \in \mathrm{Inv}(\psi) \Rightarrow \mathcal{R}(\psi^\circ(f)) = \mathcal{R}_q(\psi^\circ(f))$. It follows from (4.30) that $\psi^\circ(f) = \theta(\psi^\circ(f))$. But, since θ is increasing and ψ° is anti-extensive, we have that $\theta(\psi^\circ(f)) \leq \theta(f)$. Therefore, $\psi^\circ(f) \leq \theta(f)$. To show the converse inequality, note that, for $R \in \mathcal{R}_q(f)$, Lemma 4.2.10(a) implies that $\widetilde{\rho}(f \mid h_{R,f(R)}) \in \mathrm{Inv}(\psi)$, so that, from Proposition 2.2.6(b) and the increasing property of ψ°, $\widetilde{\rho}(f \mid h_{R,f(R)}) = \psi(\widetilde{\rho}(f \mid h_{R,f(R)})) \leq \psi^\circ(\widetilde{\rho}(f \mid h_{R,f(R)})) \leq \psi^\circ(f)$, for all $R \in \mathcal{R}_q(f)$ and, therefore, $\theta(f) \leq \psi^\circ(f)$. Hence, $\psi^\circ(f) = \theta(f)$, which shows (4.37). Q.E.D.

Note that ψ° is an operator that effectively removes all regional maxima of f below q. In addition, note that $\psi \leq \psi^\circ$, as required by Proposition 2.2.6(b).

As a direct consequence of Proposition 4.2.1 and the exposition above, we have the following result.

4.2.12 Proposition. If ψ is the operator given by (4.30), then $\mathcal{L}^\psi = \mathrm{Inv}(\psi)$ is an underlattice of $\mathrm{Fun}_u(E, \overline{\mathbb{R}})$, with supremum $\bigvee^\psi$ and infimum $\bigwedge^\psi$, given by

$$\bigvee^\psi f_\alpha = \bigvee_u f_\alpha, \tag{4.38}$$

$$\bigwedge^\psi f_\alpha = \psi^\circ\left(\bigwedge_u f_\alpha\right) = \bigvee_u \{\widetilde{\rho}(\bigwedge f_\alpha \mid h_{R,(\bigwedge f_\alpha)(R)}) \mid R \in \mathcal{R}_q(\bigwedge f_\alpha)\}. \tag{4.39}$$

□

Note that $\bigwedge^\psi f_\alpha = O$ if and only if $\bigwedge f_\alpha$ has no regional maxima at level q or above. Hence, even if the functions f_α overlap according to the pointwise infimum, they can still have zero infimum in $\mathcal{L}^\psi$, provided that the overlap is "small enough." Therefore, the infimum $\bigwedge^\psi$ is less restrictive, with respect to axiom (*iii*) of connectivity classes, than the original infimum $\bigwedge_u = \bigwedge$.

Next, we introduce a sup-generating family for lattice $\mathcal{L}^\psi$.

4.2.13 Proposition. Let $\mathcal{L}^\psi = \mathrm{Inv}(\psi)$, where ψ is the operator given by (4.30). The family

$$\mathcal{S}^\psi = \{\delta_{v,t} \mid t \geq q\} \cup \{f \in \mathrm{Fun}_u(E, \overline{\mathbb{R}}) \mid \mathcal{R}(f) = \{R\},\ f(R) = q\} \tag{4.40}$$

is sup-generating in $\mathcal{L}^\psi$. □

Proof. First, note that $\mathcal{S}^\psi \subseteq \mathcal{L}^\psi$. Let $f \in \mathrm{Fun}_u(E, \overline{\mathbb{R}})$, and consider the two functions f_1, $f_2 \in \mathcal{L}^\psi$ generated by the sets

$$F_1(t) = \begin{cases} X_t(f), & \text{if } t \leq q \\ \emptyset, & \text{if } t > q \end{cases} \quad \text{and} \quad F_2(t) = \begin{cases} X_q(f), & \text{if } t \leq q \\ X_t(f), & \text{if } t > q \end{cases}, \tag{4.41}$$

for $t \in \overline{\mathbb{R}}$, respectively. Note that $\mathcal{R}_q(f_1) = \mathcal{C}(T)$, where $T = X_q(f)$ and $\mathcal{C}(T)$ denotes the set of connected components of T. From the fact that $\mathcal{L}^\psi = \mathrm{Inv}(\psi^\circ)$, we have that $f_1 = \psi^\circ(f_1) = \bigvee_u \{\widetilde{\rho}(f_1 \mid h_{R,q}) \mid R \in \mathcal{C}(T)\}$. But, according to Lemma 4.2.10(a) and (4.40), $\widetilde{\rho}(f_1 \mid h_{R,q}) \in \mathcal{S}^\psi$, for each $R \in \mathcal{C}(T)$. Hence, f_1 is sup-generated by $\mathcal{S}^\psi$. Note also that $f_2 = \bigvee\{\delta_{v,t} \mid \delta_{v,t} \leq f_2\} = \bigvee\{\delta_{v,t} \mid \delta_{v,t} \leq f_2, t \geq q\} = \bigvee_u\{\delta_{v,t} \mid \delta_{v,t} \leq f_2, t \geq q\}$, where the last equality follows from the observation that $\bigvee_u f_\alpha = \bigvee f_\alpha$, whenever the latter is u.s.c. Therefore, f_2 is also sup-generated by $\mathcal{S}^\psi$. To conclude the proof, note that it follows from (4.41) and Propositions 2.2.11(a) and 2.2.12(b) that $X_t(f_1 \vee_u f_2) = X_t(f_1 \vee f_2) = X_t(f_1) \cup X_t(f_2) = F_1(t) \cup F_2(t) = X_t(f)$, for all $t \in \overline{\mathbb{R}}$. Hence, $f = f_1 \vee_u f_2$, so that f is sup-generated by $\mathcal{S}^\psi$. Q.E.D.

From (4.40), it is clear that the family $\mathcal{S}^\psi$ consists of all pulses of height at or above q, along with the functions in $\mathcal{L}^\psi$ that have exactly one regional maximum at level q.

We can now define the flat grayscale connectivity class $\mathcal{C}^\psi$ associated with $\mathcal{C}$, as the family in $\mathcal{L}^\psi$ given by

$$\mathcal{C}^\psi = \{f \in \mathcal{L}^\psi \mid X_t(f) \in \mathcal{C}, \text{for all } t \leq q\}. \tag{4.42}$$

Indeed, we have the following result.

4.2.14 Proposition. The family $\mathcal{C}^\psi$ in (4.42) defines a connectivity class in $\mathcal{L}^\psi$. □

Proof. First, note that the zero function is trivially in $\mathcal{C}^\psi$. In addition, by using Proposition 4.2.8(b), it is easy to see that $\mathcal{S}^\psi \subseteq \mathcal{C}^\psi$. This shows axioms (i) and (ii) of a connectivity class. To show axiom (iii), consider a family $\{f_\alpha\}$ of functions in $\mathcal{C}^\psi$ such that $\bigwedge^\psi f_\alpha \neq O$. As argued previously, this means that $\bigwedge f_\alpha$ has a regional maximum at or above level q, which implies that $X_t(\bigwedge f_\alpha) = \bigcap X_t(f_\alpha) \neq \emptyset$, for all $t \leq q$, where we have used Proposition 2.2.12(b). Since $X_t(f_\alpha) \in \mathcal{C}$, for each f_α, we have that $\bigcup X_t(f_\alpha) \in \mathcal{C}$, for all $t \leq q$. Since $\mathcal{C}$ is compatible, it follows that $\overline{\bigcup X_t(f_\alpha)} \in \mathcal{C}$, for all $t \leq q$. On the other hand, from Propositions 2.2.11(a) and 4.2.6, we have that $X_t(\bigvee_u f_\alpha) = \bigcap_{s<t} \overline{\bigcup X_t(f_\alpha)}$, for all $t \in \overline{\mathbb{R}}$. Hence, for $x \in X_t(\bigvee_u f_\alpha)$, we have that $\gamma_x(X_t(\bigvee_u f_\alpha)) = \gamma_x(\bigcap_{s<t} \overline{\bigcup X_t(f_\alpha)}) = \bigcap_{s<t} \gamma_x(\overline{\bigcup X_t(f_\alpha)}) = \bigcap_{s<t} \overline{\bigcup X_t(f_\alpha)} = X_t(\bigvee_u f_\alpha) \Rightarrow X_t(\bigvee_u f_\alpha) \in \mathcal{C}$, for all $t \leq q$, where we have used the assumption that γ_x is $\downarrow$-continuous on $\mathcal{F}(E)$ and Proposition 2.2.10. Therefore, $\bigvee_u f_\alpha \in \overline{\mathcal{C}}$, as required. Q.E.D.

Clearly, a function $f \in \mathcal{L}^\psi$ is connected, according to $\mathcal{C}^\psi$, if all of its threshold sets below level q are connected. Loosely speaking, this means that f is not allowed to have any "disconnecting dips" below level q. The parameter q reflects the "richness" of the connectivity. Smaller values of q allow more functions to be considered as being connected.

The connected components of a function $f \in \mathcal{L}^\psi$ are associated with the connected components, according to $\mathcal{C}$, of the threshold set $X_q(f)$ of f at level q. More precisely, if γ_x^ψ, $x \in \mathcal{S}^\psi$, is the connectivity opening associated with the connectivity class $\mathcal{C}^\psi$, then it

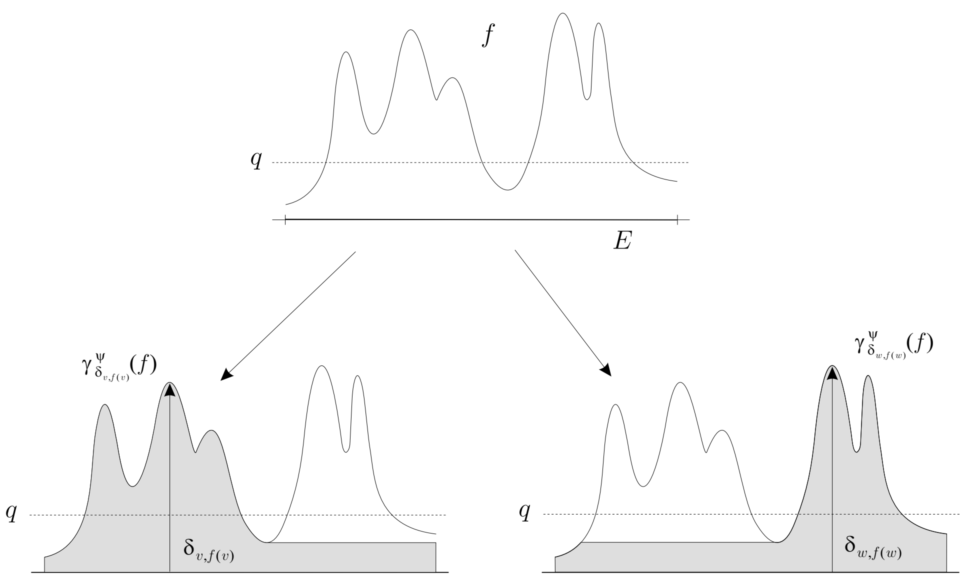


Figure 4.10: An example of flat grayscale connectivity. The function $f \in \mathcal{L}^{\psi}$ has two connected components, according to $\overline{\mathcal{C}}$. The underlying connectivity class $\mathcal{C}$ is the usual Euclidean topological connectivity on the indicated connected, closed and bounded interval E.

is not difficult to see that

$$\gamma_x^{\psi}(f) = \bigvee\nolimits_u \{\widetilde{\rho}(f \mid h_{R,f(R)}) \mid R \in \mathcal{R}_q(f) \cap \mathcal{M}_*(K(x))\}, \tag{4.43}$$

where $K(x) = \gamma_{\{v\}}(X_q(f))$, if $x = \delta_{v,t}$, $t \geq q$, or $K(x) = R$, if $x = f \in \mathrm{Fun}_u(E, \overline{\mathbb{R}})$, such that $\mathcal{R}(f) = \{R\}$, $f(R) = q$. In other words, $\gamma_x^{\psi}(f)$ preserves the regional maxima of f that are contained in the connected component of $X_q(f)$ marked by the sup-generator x. Clearly, the number of connected components of f, according to $\mathcal{C}^{\psi}$, equals the number of binary connected components of $X_q(f)$. Fig. 4.10 depicts the connected components of a function $f \in \mathcal{L}^{\psi}$ according to flat grayscale connectivity. The underlying connectivity class $\mathcal{C}$ is the usual Euclidean topological connectivity on the indicated connected, closed and bounded interval E.

The notion of flat grayscale connectivity applies to the discrete case as well. The main ideas apply to this case, and certain assumptions are substantially simplified. More specifically, consider a finite subset $E \subseteq \mathbb{Z}^n$, and the lattice $\mathcal{L} = \mathrm{Fun}(E, \mathcal{T})$ of all discrete functions from E into $\mathcal{T}$, where $\mathcal{T} = \overline{\mathbb{Z}}$ or $\mathcal{T} = \{0, 1, \ldots, R-1\}$, for a finite integer $R \geq 2$. Take the underlying binary connectivity class $\mathcal{C}$ to be *any* strong connectivity class $\mathcal{C} \subseteq \mathcal{P}(E)$. A little thought reveals that all relevant results given earlier hold in this case.

In practice, the discrete version of flat grayscale connectivity offers a potentially useful tool for image segmentation. Figs. 4.11 and 4.12 depict two segmentation examples. Note that the original images are first projected into lattice $\mathcal{L}^{\psi}$ by applying the characteristic opening ψ°, given by (4.37). Note that the projected images are good approximations of the originals. The image depicted in Fig. 4.11 contains several clusters of cornea cells. Despite being a noisy image, the three "largest" grayscale connected components (the connected components with the largest, in volume, subgraphs) provide a good segmentation of the three main cell clusters. In Fig. 4.12, we depict four connected components of the Lenna image, labelled as "hat highlight," "shoulder highlight," "right eye," and "background object," which correspond to a highlight on the hat, a highlight on the shoulder, the right eye, and a bright object in the background, respectively.

4.3 Second-Generation Connectivity

A second-generation connectivity class is a new connectivity class generated from an existing one by means of a suitably defined operator [77–80]. In this section, we study two classes of second-generation connectivity on complete lattices. The first one is based on clustering elements of the original connectivity by means of a clustering operator. We give an axiomatic formulation of clustering operators, and present a new example of a clustering connectivity class based on morphological sampling operators. The second class is the dual, in a sense, of the first. It is based on restricting a given connectivity class by means of a contraction operator. This includes the case of a connectivity class restricted by openings, previously studied for the binary case in [65, 66], which we generalize to atomic lattices.

4.3.1 Connectivity Based on Clustering

The basic idea pursued here is grouping the connected components of a given object into "clusters." One way to do this is by means of an operator that "joins" the connected components to be clustered. More precisely, let $\mathcal{L}$ be a lattice and $\mathcal{C}$ be a connectivity class in $\mathcal{L}$. We say that $A \in \mathcal{L}$ is a cluster if $\psi(A) \in \mathcal{C}$, for some suitably chosen operator ψ on $\mathcal{L}$. In addition, we say that A is composed of clusters $A \wedge C$, where $C \lessdot \psi(A)$. See Fig. 4.13 for an illustration.

Next, we list intuitively desirable properties that a clustering operator ψ might possess.

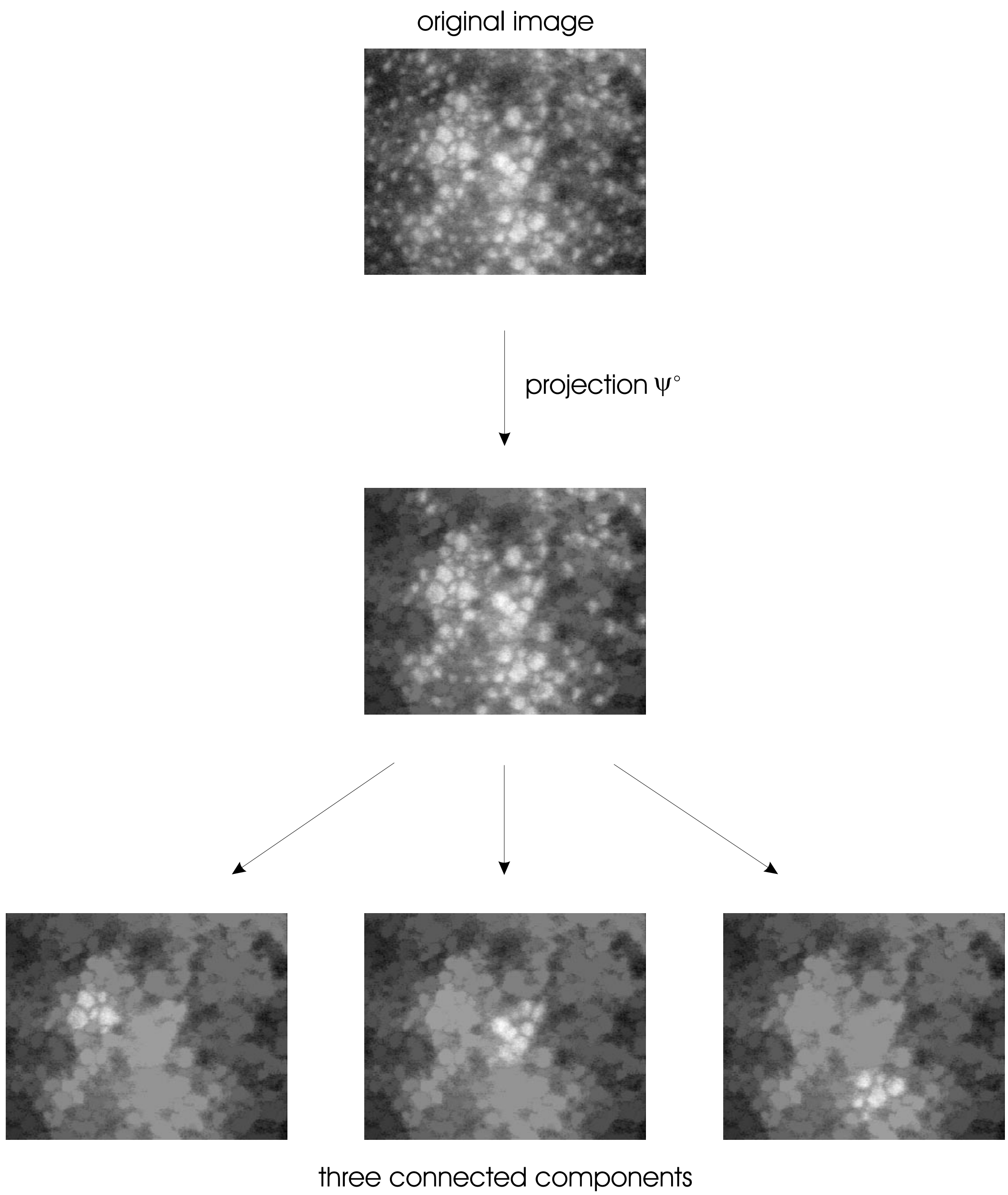


Figure 4.11: An example of segmentation of a cornea cells image by using flat grayscale connectivity. Only the three "largest" connected components are depicted. In this example, the image has 256 gray levels and $q = 158$. The underlying connectivity class $\mathcal{C}$ is the usual 8-adjacency connectivity.

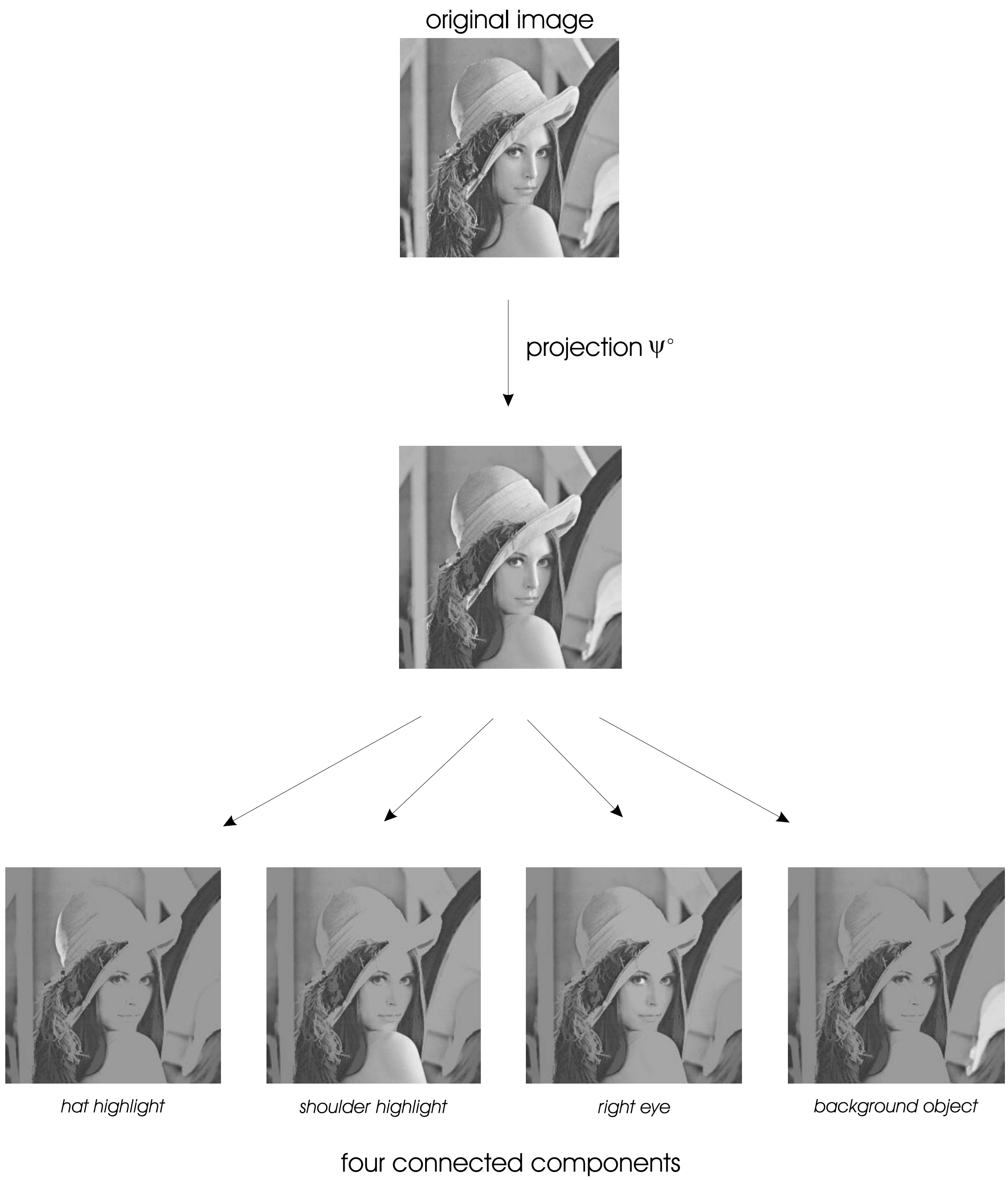


Figure 4.12: An example of segmentation of the Lenna image by using flat grayscale connectivity. These connected components correspond to clearly identifiable regions, labelled as “hat highlight,” “shoulder highlight,” “right eye,” and “background object.” In this example, the image has 256 gray levels and $q = 220$. The underlying connectivity class $\mathcal{C}$ is the usual 8-adjacency connectivity.

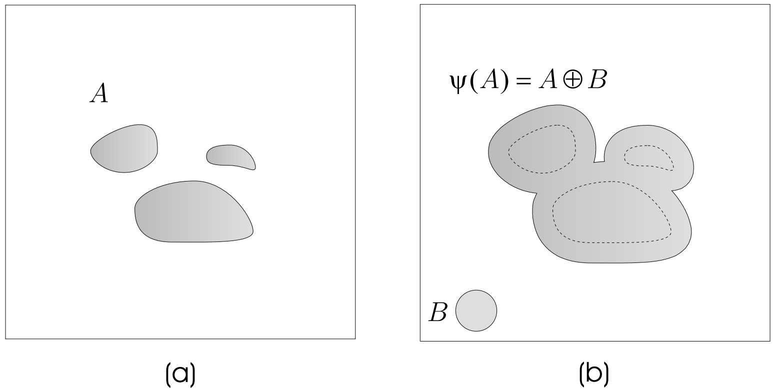


Figure 4.13: (a) A subset A of the 2-D Euclidean space that contains three connected components. (b) The set $\psi(A) = A \oplus B$ is connected, so that A is a cluster. The set A is composed of the single cluster $A \cap \psi(A)$.

4.3.1 Condition.

(*i*) ψ is increasing and extensive.

(*ii*) ψ is *connectivity-preserving*; i.e., $\psi(\mathcal{C}) \subseteq \mathcal{C}$.

(*iii*) If $\psi(A_\alpha) \in \mathcal{C}$, for all α, and $\bigwedge A_\alpha \neq O$, then $\psi(\bigvee A_\alpha) \in \mathcal{C}$.

(*iv*) ψ does not create connected components; i.e., $C \lessdot \psi(A) \Rightarrow A \wedge C \neq O$.

(*v*) ψ treats the clusters of A independently; i.e., $C \lessdot \psi(A) \Rightarrow \psi(A \wedge C) = C$. △

Item (*i*) implies that $\psi(A)$ joins connected components of A (ψ is extensive) in a way that preserves ordering (ψ is increasing). Item (*ii*) means that the only cluster of a connected A is A itself. In loose terms, item (*iii*) means that union of intersecting clusters is a cluster as well. Item (*iv*) is self-evident, whereas item (*v*) implies that $A \wedge C$ defines a cluster, for all $C \lessdot \psi(A)$. Since $\psi(A) = \bigvee_{C \lessdot \psi(A)} C$, item (*v*) also implies that $\psi(A) = \bigvee_{C \lessdot \psi(A)} \psi(A \wedge C)$; i.e., $\psi(A)$ can be computed cluster by cluster.

4.3.2 Definition. Let $\mathcal{L}$ be a lattice with sup-generating family $\mathcal{S}$, furnished with a connectivity class $\mathcal{C}$. An increasing and extensive operator ψ on $\mathcal{L}$ is said to be a *strong clustering* if $\psi(O) = O$ and

$$\psi(\mathbf{id} \wedge \gamma_x \psi) = \gamma_x \psi, \quad \text{for all } x \in \mathcal{S}. \tag{4.44}$$

△

4.3.3 Proposition. An operator ψ on $\mathcal{L}$ satisfies all items of Condition 4.3.1 if and only if ψ is a strong clustering. $\square$

PROOF. "$\Leftarrow$": Note that, by definition, ψ satisfies items (i) and (v) of Condition 4.3.1. To show item (iv), let $x \leq \psi(A)$ such that $A \wedge \gamma_x\psi(A) = O$. Then, (4.44) implies that $\psi(O) = \gamma_x\psi(A) \neq O$, which is a contradiction. Item (ii) follows from the observation that, since $A \in \mathcal{C}$ and $A \leq \psi(A)$, A must be contained in one of the connected components of $\psi(A)$, which, from item (iv), must be the only one. This means that $\psi(A) \in \mathcal{C}$, so that ψ is connectivity-preserving. To show item (iii), pick a sup-generator $x \leq \bigwedge A_\alpha$. We have that $x \leq A_\alpha \leq \psi(A_\alpha)$, for all α. Note that $x \leq \psi(A_\alpha) \in \mathcal{C}$ and $A_\alpha \leq \bigvee A_\alpha \Rightarrow \psi(A_\alpha) \leq \psi(\bigvee A_\alpha)$, for all α, since ψ is increasing. It follows that $\psi(A_\alpha) \leq \gamma_x\psi(\bigvee A_\alpha)$, for all α. Hence, $\bigvee A_\alpha \leq \bigvee \psi(A_\alpha) \leq \gamma_x\psi(\bigvee A_\alpha)$, where the first inequality follows from the extensivity of ψ. Since ψ does not create connected components, this means that $\gamma_x\psi(\bigvee A_\alpha)$ is the only connected component of $\psi(\bigvee A_\alpha)$; i.e., $\psi(\bigvee A_\alpha) \in \mathcal{C}$.

"$\Rightarrow$": From item (i), ψ is increasing and extensive. We show that $\psi(O) = O$. Suppose that $\psi(O) \neq O$. Then, we can pick a nonzero $C \lessdot \psi(O)$, in which case item (iv) implies that $O \wedge C \neq O$, which is a contradiction. To verify (4.44), consider an $A \in \mathcal{L}$. If $x \not\leq \psi(A)$, then (4.44) holds, since $\psi(O) = O$, whereas if $x \leq \psi(A)$, (4.44) follows from item (v). Q.E.D.

By relaxing the requirements on a strong clustering, we introduce the notion of a clustering.

4.3.4 Definition. An operator ψ on $\mathcal{L}$ that satisfies items (i)–(iii) of Condition 4.3.1 is said to be a *weak clustering*, or simply a *clustering*. $\triangle$

Clearly, every strong clustering is a clustering. However, a clustering may in general create connected components and does not have to treat clusters independently.

The following proposition shows that, starting from a given connectivity class, a clustering generates a new connectivity class, which produces a coarser PCC, as expected from a clustering operation.

4.3.5 Proposition. Let $\mathcal{L}$ be a lattice with sup-generating family $\mathcal{S}$, furnished with a connectivity class $\mathcal{C}$, and let ψ be a clustering on $\mathcal{L}$.

(a) The family

$$\mathcal{C}^\psi = \psi^{-1}(\mathcal{C}) = \{A \in \mathcal{L} \mid \psi(A) \in \mathcal{C}\} \tag{4.45}$$

is a connectivity class in $\mathcal{L}$.

(b) For $A \in \mathcal{L}$, the PCC of A according to $\mathcal{C}^{\psi}$ is coarser than the PCC of A according to $\mathcal{C}$; i.e.,

$$c_A(x) = \gamma_x(A) \leq \gamma_x^{\psi}(A) = c_A^{\psi}(x), \quad \text{for all } x \leq A, \tag{4.46}$$

where $\{\gamma_x^{\psi} \mid x \in \mathcal{S}\}$ are the connectivity openings associated with $\mathcal{C}^{\psi}$. □

PROOF. (a): Note that $O \in \mathcal{C}^{\psi}$ and $\mathcal{S} \subseteq \mathcal{C}^{\psi}$, since ψ is connectivity-preserving. Item (*iii*) of Condition 4.3.1 means that, if $A_\alpha \in \mathcal{C}^{\psi}$ with $\bigwedge A_\alpha \neq O$, then $\bigvee A_\alpha \in \mathcal{C}^{\psi}$. Hence, $\mathcal{C}^{\psi}$ satisfies all the axioms of a connectivity class.

(b): Note that, since ψ is connectivity-preserving, we have that $\mathcal{C} \supseteq \psi(\mathcal{C}) \Rightarrow \mathcal{C}^{\psi} = \psi^{-1}(\mathcal{C}) \supseteq \psi^{-1}(\psi(\mathcal{C})) \supseteq \mathcal{C}$. Hence, $\mathcal{C} \subseteq \mathcal{C}^{\psi}$, so that $\gamma_x(A) \leq \gamma_x^{\psi}(A)$, for all $x \leq A$. Q.E.D.

The previous result says that the family of all clusters in $\mathcal{L}$ is a connectivity class. Moreover, the PCC of $A \in \mathcal{L}$, according to the new connectivity class $\mathcal{C}^{\psi}$, is coarser than the PCC of A, according to the original connectivity class $\mathcal{C}$. This is equivalent to the fact that $\mathcal{C}^{\psi}$ is *richer* than $\mathcal{C}$; i.e., $\mathcal{C}^{\psi} \supseteq \mathcal{C}$. We refer to $\mathcal{C}^{\psi}$ as a *clustering-based connectivity class*.

We remark that the increasing and extensivity properties of ψ are not required for the proof of Proposition 4.3.5. Moreover, for part (a), it is only necessary that ψ be connectivity-preserving on the zero element and the sup-generators. However, a non-increasing and non-extensive operator that does not preserve connectivity over all $\mathcal{L}$ can be hardly considered to be a clustering.

In order to characterize the connected components associated with $\mathcal{C}^{\psi}$, we need to assume that ψ is a strong clustering. This is clear from the following proposition.

4.3.6 Proposition. Let $\mathcal{L}$ be a lattice with sup-generating family $\mathcal{S}$, furnished with a connectivity class $\mathcal{C}$. Let $\{\gamma_x \mid x \in \mathcal{S}\}$ and ρ be the connectivity openings and reconstruction operator, respectively, associated with $\mathcal{C}$. If ψ is a strong clustering on $\mathcal{L}$, then:

(a) The connectivity openings $\{\gamma_x^{\psi} \mid x \in \mathcal{S}\}$ associated with $\mathcal{C}^{\psi}$ are given by

$$\gamma_x^{\psi}(A) = \begin{cases} A \wedge \gamma_x \psi(A), & \text{if } \; x \leq A \\ O, & \text{if } \; x \not\leq A \end{cases}, \quad \text{for } x \in \mathcal{S}, \tag{4.47}$$

for $A \in \mathcal{L}$.

(b) If $\mathcal{L}$ is infinite $\vee$-distributive, the reconstruction operator ρ^{ψ} associated with $\mathcal{C}^{\psi}$ is given by

$$\rho^{\psi}(A \mid M) = A \wedge \rho(\psi(A) \mid A \wedge M), \tag{4.48}$$

for $A, M \in \mathcal{L}$. □

Proof. (a): If $x \not\leq A$, then there is nothing to prove; therefore, assume that $x \leq A$. From the extensivity of ψ, this implies that $x \leq \psi(A) \Rightarrow x \leq \gamma_x\psi(A) \Rightarrow x \leq A \wedge \gamma_x\psi(A)$. In addition, (4.44) gives $\psi(A \wedge \gamma_x\psi(A)) \in \mathcal{C} \Rightarrow A \wedge \gamma_x\psi(A) \in \mathcal{C}^{\psi}$. Hence, $\gamma_x^{\psi}(A) \geq \gamma_x^{\psi}(A \wedge \gamma_x\psi(A)) = A \wedge \gamma_x\psi(A)$. To show the reverse inequality, let $C \in \mathcal{C}^{\psi}$ such that $x \leq C \leq A$. We have that $\psi(C) \in \mathcal{C}$. In addition, $x \leq C \leq \psi(C)$ and $\psi(C) \leq \psi(A)$, by the extensivity and increasing properties of ψ. Therefore, $\psi(C) = \gamma_x\psi(C) \leq \gamma_x\psi(A) \Rightarrow C \leq A \wedge \gamma_x\psi(A)$. It then follows from (4.8) that $\gamma_x^{\psi}(A) \leq A \wedge \gamma_x\psi(A)$.

(b): From part (a), we have that $\rho^{\psi}(A \mid M) = \rho^{\psi}(A \mid A \wedge M) = \bigvee_{x \leq A \wedge M} \gamma_x^{\psi}(A) = \bigvee_{x \leq A \wedge M} A \wedge \gamma_x\psi(A)$. By the infinite $\vee$-distributivity of $\mathcal{L}$, $\rho^{\psi}(A \mid M) = A \wedge \bigvee_{x \leq A \wedge M} \gamma_x\psi(A)$ $= A \wedge \rho(\psi(A) \mid A \wedge M)$. Q.E.D.

The previous result says that, in the case of a strong clustering ψ, the connected components of A, according to the clustering-based connectivity class $\mathcal{C}^{\psi}$, correspond to the clusters $A \wedge C$, for $C \lessdot \psi(A)$. Therefore, the PCC of A according to $\mathcal{C}^{\psi}$ corresponds to the segmentation of A into its clusters. It follows that $A = \bigvee_{C \lessdot \psi(A)} A \wedge C$; i.e., A is "composed" of its clusters. In addition, if $\mathcal{L}$ is infinite $\vee$-distributive, ρ^{ψ} recovers the clusters of A that are marked by $M \wedge A$. Note that, when $M = x \leq A$, in which case $\rho(A \mid M) = \gamma_x(A)$, (4.47) and (4.48) are in agreement. In general, these properties are not valid when ψ is only a weak clustering.

When ψ is an $\downarrow$-continuous strong clustering on $\mathcal{P}(E)$, there is a useful characterization of the grayscale reconstruction operator $\widetilde{\rho}^{\psi}$ generated by ρ^{ψ}. This is shown by the following proposition.

4.3.7 Proposition. Let $\mathcal{C}$ be a connectivity class in a lattice $\mathcal{P}(E)$ and $\widetilde{\rho}$ be the associated grayscale reconstruction operator on $\mathrm{Fun}(E, \overline{\mathbb{R}})$, given by (4.21), with $\mathcal{T} = \overline{\mathbb{R}}$. If ψ is an $\downarrow$-continuous strong clustering on $\mathcal{P}(E)$, the grayscale reconstruction operator $\widetilde{\rho}^{\psi}$ on $\mathrm{Fun}(E, \overline{\mathbb{R}})$ associated with $\mathcal{C}^{\psi}$ is given by

$$\widetilde{\rho}^{\psi}(f \mid g) = f \wedge \widetilde{\rho}(\overline{\psi}(f) \mid f \wedge g), \tag{4.49}$$

where $\overline{\psi}$ is the flat operator generated by ψ (see Section 2.2). □

PROOF. Let ρ and ρ^ψ be the reconstruction operators associated with $\mathcal{C}$ and $\mathcal{C}^\psi$, respectively. From Section 2.2, recall that $\downarrow$-continuity of ψ implies that $X_t(\overline{\psi}(f)) = \psi(X_t(f))$, for all $t \in \overline{\mathbb{R}}$. By using Proposition 4.3.6(b), we have that

$$\begin{aligned}\widetilde{\rho}^{\psi}(f \mid g)(v) &= \bigvee\{t \in \overline{\mathbb{R}} \mid v \in \rho^{\psi}(X_t(f) \mid X_t(g))\} \\ &= \bigvee\{t \in \overline{\mathbb{R}} \mid v \in X_t(f) \cap \rho(\psi(X_t(f)) \mid X_t(f) \cap X_t(g))\} \\ &= \bigvee\{t \in \overline{\mathbb{R}} \mid v \in X_t(f)\} \wedge \bigvee\{t \in \overline{\mathbb{R}} \mid v \in \rho(X_t(\overline{\psi}(f)) \mid X_t(f \wedge g))\} \\ &= f(v) \wedge \widetilde{\rho}(\overline{\psi}(f) \mid f \wedge g)(v), \end{aligned} \tag{4.50}$$

for all $v \in E$, which shows (4.49). Q.E.D.

The prototypical examples of clusterings are given by connectivity-preserving closings and connectivity-preserving extensive dilations. This is shown next.

4.3.8 Proposition. Let $\mathcal{L}$ be a lattice furnished with a connectivity class $\mathcal{C}$. A connectivity-preserving closing ϕ on $\mathcal{L}$ is a clustering. $\square$

PROOF. Items (i) and (ii) of Condition 4.3.1 are satisfied by definition. To show that item (iii) is also satisfied, consider a family $\{A_\alpha\}$ in $\mathcal{L}$ such that $\phi(A_\alpha) \in \mathcal{C}$, for all α, and $\bigwedge A_\alpha \neq O$. By the extensivity of ϕ, this implies that $\bigwedge \phi(A_\alpha) \geq \bigwedge A_\alpha \neq O$, so that $\bigvee \phi(A_\alpha) \in \mathcal{C}$, from axiom (iii) of connectivity classes. We show that $\phi(\bigvee A_\alpha) = \phi(\bigvee \phi(A_\alpha))$, which implies that $\phi(\bigvee A_\alpha) \in \mathcal{C}$, since ϕ is connectivity-preserving. By the extensivity of ϕ, we have that $\bigvee \phi(A_\alpha) \geq \bigvee A_\alpha$, so that $\phi(\bigvee \phi(A_\alpha)) \geq \phi(\bigvee A_\alpha)$, since ϕ is increasing. On the other hand, and since ϕ is increasing, we have that $\bigvee \phi(A_\alpha) \leq \phi(\bigvee A_\alpha)$, so that $\phi(\bigvee \phi(A_\alpha)) \leq \phi\phi(\bigvee A_\alpha) = \phi(\bigvee A_\alpha)$, by the idempotence of ϕ. Hence, $\phi(\bigvee A_\alpha) = \phi(\bigvee \phi(A_\alpha))$, as required. Q.E.D.

4.3.9 Proposition. Let $\mathcal{L}$ be a lattice with sup-generating family $\mathcal{S}$, furnished with a connectivity class $\mathcal{C}$. Let δ be an extensive dilation on $\mathcal{L}$ such that $\delta(x) \in \mathcal{C}$, for every $x \in \mathcal{S}$.

(a) δ is a clustering on $\mathcal{L}$.

(b) If $\mathcal{L}$ is infinite $\vee$-distributive, then δ is a strong clustering on $\mathcal{L}$. $\square$

PROOF. (a): Item (i) of Condition 4.3.1 is satisfied by definition. We show that δ is connectivity-preserving. Note that $\delta(O) = O$, so that $\delta(O) \in \mathcal{C}$. Let $A \in \mathcal{C} \setminus \{O\}$. Note that $\delta(A) = \delta(\bigvee_{x \in \mathcal{S}(A)} x) = \bigvee_{x \in \mathcal{S}(A)} \delta(x)$, where $\mathcal{S}(A) = \{x \in \mathcal{S} \mid x \leq A\}$. Since δ is extensive, it follows that $\delta(A) = A \vee \delta(A) = A \vee \bigvee_{x \in \mathcal{S}(A)} \delta(x) = \bigvee_{x \in \mathcal{S}(A)} [A \vee \delta(x)]$. But $A, \delta(x) \in \mathcal{C}$, for $x \in \mathcal{S}(A) \subseteq \mathcal{S}$, with $A \wedge \delta(x) \geq x \neq O$, so that $A \vee \delta(x) \in \mathcal{C}$, for all $x \in \mathcal{S}(A)$. In addition, $\bigwedge_{x \in \mathcal{S}(A)} [A \vee \delta(x)] \geq A \neq O$, which implies that $\delta(A) = \bigvee_{x \in \mathcal{S}(A)} [A \vee \delta(x)] \in \mathcal{C}$, as required. Finally, to show item (*iii*) of Condition 4.3.1, let $\{A_\alpha\}$ be a family in $\mathcal{L}$ such that $\delta(A_\alpha) \in \mathcal{C}$, for all α, and $\bigwedge A_\alpha \neq O$. We have that $\bigwedge \delta(A_\alpha) \geq \bigwedge A_\alpha \neq O$, so that $\delta(\bigvee A_\alpha) = \bigvee \delta(A_\alpha) \in \mathcal{C}$.

(b): Since δ is increasing, extensive and $\delta(O) = O$, to show that δ is a strong clustering on $\mathcal{L}$, we only need to show that δ satisfies (4.44), i.e.,

$$\delta(A \wedge \gamma_x \delta(A)) = \gamma_x \delta(A), \quad \text{for all } x \in \mathcal{S}, A \in \mathcal{L}. \tag{4.51}$$

If $x \not\leq \delta(A)$, the result is trivial. So, let $x \leq \delta(A)$. Equation (4.51) then reduces to $\delta(A \wedge C) = C$, for all $C \lessdot \delta(A)$. In the following, we use the notation $C \lessdot^{\delta} A$ to indicate that C is a connected component of A according to the connectivity class $\mathcal{C}^{\delta}$.

First, we show that, for a given $C \lessdot \delta(A)$, we have $C = \delta(C_0)$, for some $C_0 \lessdot^{\delta} A$. Clearly, $C \leq \delta(A) \Rightarrow C = C \wedge \delta(A) = C \wedge \delta(\bigvee\{C' \mid C' \lessdot^{\delta} A\}) = C \wedge \bigvee\{\delta(C') \mid C' \lessdot^{\delta} A\} = \bigvee\{C \wedge \delta(C') \mid C' \lessdot^{\delta} A\}$, from the infinite $\vee$-distributivity of $\mathcal{L}$. Note that $\delta(C') \in \mathcal{C}$ and $\delta(C') \leq \delta(A)$, for all $C' \lessdot^{\delta} A$. From the maximality property of connected components (see Definition 4.1.5), this implies that either $\delta(C') \leq C$ or $C \wedge \delta(C') = O$. We can then write $C = \bigvee\{\delta(C') \mid C' \lessdot^{\delta} A, \delta(C') \leq C\} = \delta(K)$, where $K = \bigvee\{C' \mid C' \lessdot^{\delta} A, \delta(C') \leq C\}$. Note that this means that $K \in \mathcal{C}^{\delta}$. Again, from the maximality property of connected components, this implies that the family $\{C' \mid C' \lessdot^{\delta} A, \delta(C') \leq C\}$ consists of a single connected component $C_0 \lessdot^{\delta} A$, so that $K = C_0$ and $C = \delta(C_0)$.

To arrive at the desired result, it suffices to show that $C_0 = A \wedge \delta(C_0)$ (this can be shown without assuming infinite $\vee$-distributivity for $\mathcal{L}$, although our proof uses this assumption). We have that $A \wedge \delta(C_0) = \bigvee\{C' \mid C' \lessdot^{\delta} A\} \wedge \delta(C_0) = \bigvee\{C' \wedge \delta(C_0) \mid C' \lessdot^{\delta} A\}$, from the infinite $\vee$-distributivity of $\mathcal{L}$. Now, we claim that $C' \wedge \delta(C_0) \neq O \Rightarrow C' = C_0$, for all $C' \lessdot^{\delta} A$. This follows from the fact that, as shown in part (a), δ is a clustering, so that $\delta(C'), \delta(C_0) \in \mathcal{C}$ and $\delta(C') \wedge \delta(C_0) \geq C' \wedge \delta(C_0) \neq O$ imply that $\delta(C' \vee C_0) \in \mathcal{C} \Rightarrow C' \vee C_0 \in \mathcal{C}^{\delta} \Rightarrow C' = C_0$, from the maximality property of connected components. Hence, $A \wedge \delta(C_0) = C_0 \wedge \delta(C_0) = \delta(C_0)$. Q.E.D.

The previous proof is partially inspired by the proofs of similar results found in [79]. The infinite $\vee$-distributivity of $\mathcal{L}$ is essential in part (b). In fact, J. Serra gives an example of an extensive dilation that is not a strong clustering on the lattice $\mathcal{L} = \text{Fun}_u(\mathbb{R}, [0,1])$ of u.s.c. functions from $\mathbb{R}$ into $[0,1]$, which is not infinite $\vee$-distributive [79].

Clustering-based connectivity classes generated from dilations and closings are referred to as *dilation-based connectivity classes* and *closing-based connectivity classes*, respectively.

In the following, we discuss specific examples of clustering-based connectivity classes.

4.3.10 Example.

(a) Consider the lattice $\mathcal{L} = \mathcal{P}(E)$ of all subsets of a topological space E, with the points as sup-generators, and let $\mathcal{C}$ be the family of topologically connected sets in E. Consider the closing $\phi(A) = \overline{A}$, $A \in \mathcal{P}(E)$. From Proposition 3.1.3(b), ϕ is connectivity-preserving and is thus a clustering (see Proposition 4.3.8). In particular, $\mathcal{C}^\phi = \{A \in \mathcal{P}(E) \mid \overline{A} \in \mathcal{C}\}$ is a closing-based connectivity class. However, ϕ can create connected components. For instance, let $A = \{1/n \mid n \in \mathbb{Z}_+\}$. Then, $\phi(A) = \overline{A} = \{0\} \cup A$. But $\{0\}$ is a connected component of $\phi(A)$ that does not touch A. Since ϕ creates connected components, it is not a strong clustering.

(b) Consider either the lattice $\mathcal{L} = \mathcal{P}(\mathbb{R}^n)$ or the lattice $\mathcal{L} = \mathcal{P}(\mathbb{Z}^n)$, with the points as sup-generators, and let $\mathcal{C}$ be a translation-invariant connectivity class in $\mathcal{L}$. Consider the dilation $\delta(A) = A \oplus B$ on $\mathcal{L}$, where $B \in \mathcal{C}$ and B contains the origin of $\mathbb{R}^n$ or $\mathbb{Z}^n$. It is easy to see that δ satisfies all conditions of Proposition 4.3.9. In particular, $\mathcal{C}^\delta = \{A \in \mathcal{L} \mid A \oplus B \in \mathcal{C}\}$ is a dilation-based connectivity class. Since $\mathcal{P}(\mathbb{R}^n)$ and $\mathcal{P}(\mathbb{Z}^n)$ are infinite $\vee$-distributive, δ is a strong clustering in both cases. See Fig. 4.14 for a particular example, where $\mathcal{L} = \mathcal{P}(\mathbb{R}^2)$, $\mathcal{C}$ is the family of topologically connected sets in $\mathcal{P}(\mathbb{R}^2)$, and B is a Euclidean disk that contains the origin of $\mathbb{R}^2$. Note that the PCC of A and the reconstruction of A from a marker M, according to $\mathcal{C}^\delta$, can be easily computed using (4.47) and (4.48). On the other hand, Fig. 4.15 illustrates the grayscale reconstruction operator $\widetilde{\rho}^\delta$ on $\text{Fun}(\mathbb{R}, \overline{\mathbb{R}})$, associated with $\mathcal{C}^\delta$, when $\mathcal{L} = \mathcal{P}(\mathbb{R})$, $\mathcal{C}$ is the family of topologically connected sets in $\mathcal{P}(\mathbb{R})$, and B is a line structuring element.

(c) Consider the lattice $\mathcal{L} = \mathcal{P}(\mathbb{R}^n)$, with the points as sup-generators, and let $\mathcal{C}$ be the family of topologically path-connected subsets of $\mathbb{R}^n$, with the usual topology. Consider the *line closing* $\phi_L(A) = A \bullet L$, where $L = L(x, y)$ is the line segment joining the points $x, y \in \mathbb{R}^n$. It is easy to see that any point $v \in A \bullet L$ must lie in a line

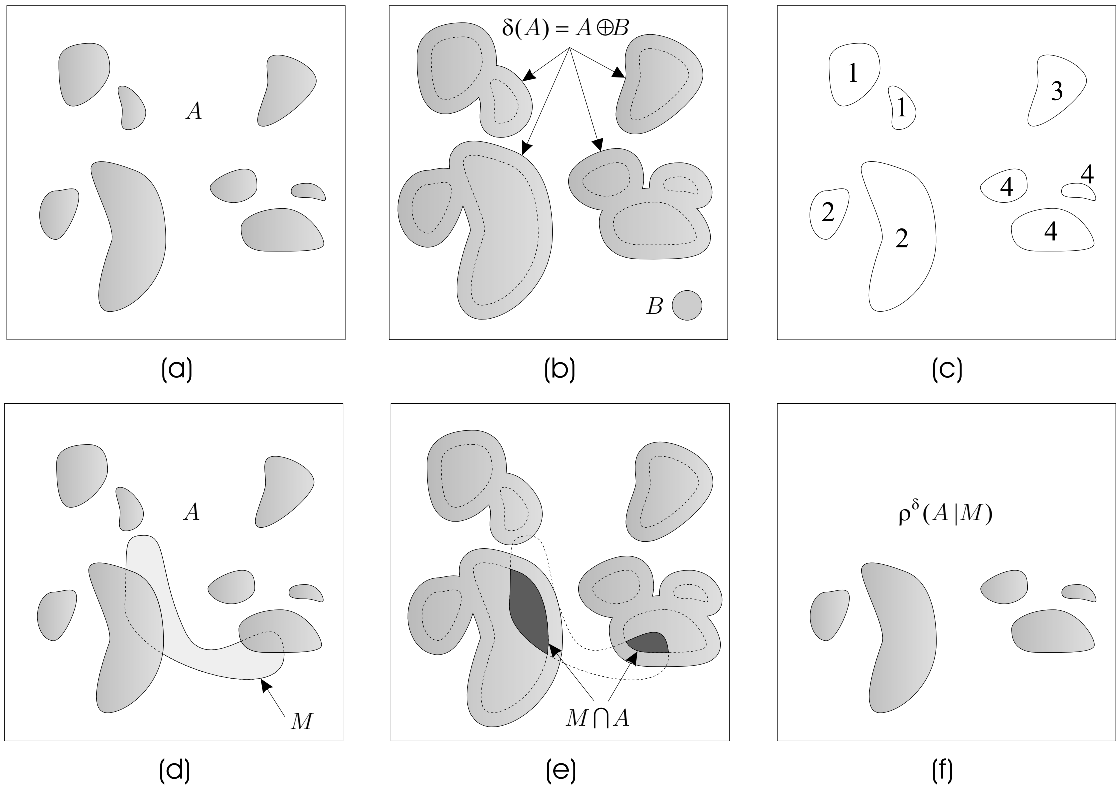


Figure 4.14: An example of a dilation-based connectivity class $\mathcal{C}^\delta$. (a) A subset A of the 2-D Euclidean space. (b) The dilation $\delta(A) = A \oplus B$. (c) The PCC of A according to $\mathcal{C}^\delta$. Note that $\gamma_x^\delta(A) = A \cap \gamma_x(A \oplus B)$. (d) A marker M superimposed on A. (e) The marker $M \cap A$ superimposed on $\delta(A) = A \oplus B$. (f) The reconstruction $\rho^\delta(A \mid M)$ according to $\mathcal{C}^\delta$. Note that $\rho^\delta(A \mid M) = A \cap \rho(A \oplus B \mid M \cap A)$, and that the cluster at the top-left corner of (a) is not part of the reconstruction, even though the dilation of that cluster intersects M. It is instructive to compare the result in (f) to that of Fig. 4.6(b).

segment $L'(v, w)$, with $w \in A$, such that $L' \subseteq A \bullet L$. Using this characterization, it is easy to show that ϕ_L is connectivity-preserving; hence, ϕ_L is a clustering. In particular, $\mathcal{C}^{\phi_L} = \{A \in \mathcal{P}(\mathbb{R}^n) \mid \phi_L(A) \in \mathcal{C}\}$ is a closing-based connectivity class. $\Diamond$

Contrary to what might be expected, and despite Example 4.3.10(c) above, structural closings $\phi_B(A) = A \bullet B$ are not in general connectivity-preserving, even if B is connected. See Fig. 4.16 for an illustration (this example is similar to the one in [34, Fig. 9.19]). In addition, the intersection of connectivity-preserving closings may not be connected. For example, consider the lattice $\mathcal{L} = \mathcal{P}(\mathbb{R}^2)$, furnished with the Euclidean topological connectivity, and let ϕ be the intersection of two structural line closings: $\phi(A) = (A \bullet L_1) \cap (A \bullet L_2)$,

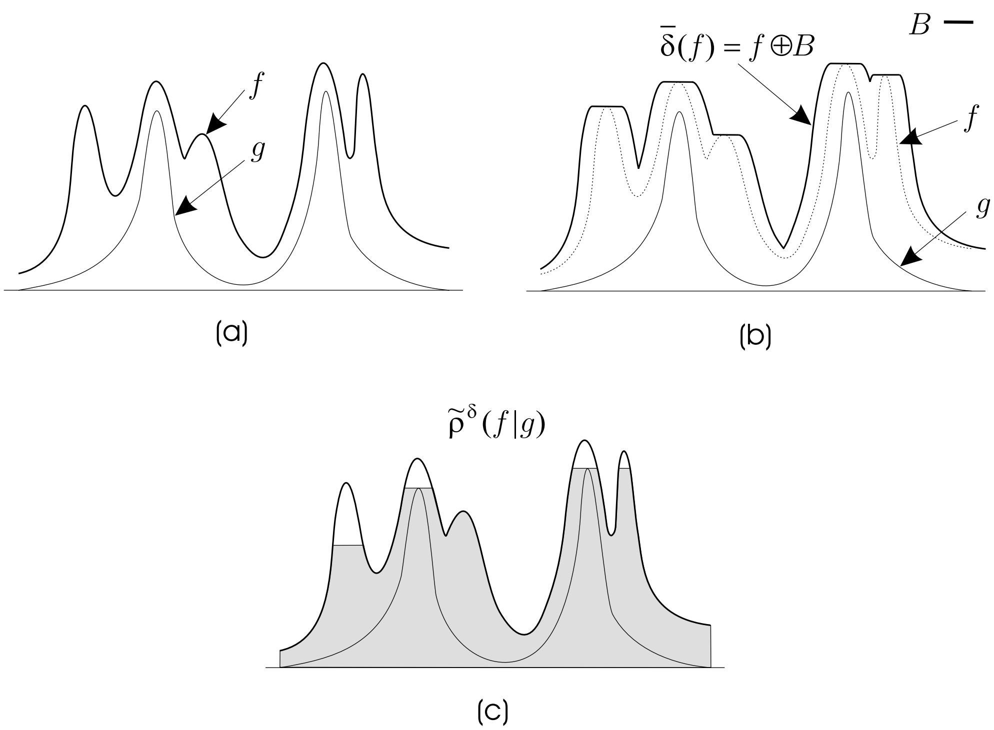


Figure 4.15: An example of grayscale reconstruction associated with a dilation-based connectivity class $\mathcal{C}^{\delta}$, generated by means of a dilation $\delta(A) = A \oplus B$, where B is the indicated line structuring element. (a) Original image f and a marker g. (b) The flat grayscale dilation $\overline{\delta}(f) = f \oplus B$. (c) The grayscale reconstruction $\widetilde{\rho}^{\delta}(f \mid g)$, according to $\mathcal{C}^{\delta}$. The original connectivity class $\mathcal{C}$ is the usual topological connectivity of the real line. It is instructive to compare the result in (c) to that of Fig. 4.7(b).

where L_1 and L_2 are the two perpendicular line structuring elements depicted in Fig. 4.17. As we have discussed in Example 4.3.10(c), $\phi_{L_1}(A) = A \bullet L_1$ and $\phi_{L_2}(A) = A \bullet L_2$ are connectivity-preserving; however, $\phi = \phi_{L_1} \wedge \phi_{L_2}$ is not.

An interesting example of a clustering-based connectivity class is provided by operators associated with the theory of *morphological sampling*. In the following, we outline some basic aspects of this theory. For a more general treatment, the reader is referred to [33, 34, 83].

Consider the lattice $\mathcal{L} = \mathcal{P}(\mathbb{R}^n)$, with the points as sup-generators. Let $S \subset \mathbb{R}^n$ be a regular grid in $\mathbb{R}^n$, given by $S = \{k_1 u_1 + \cdots + k_n u_n \mid k_i \in \mathbb{Z}\}$, where $\{u_i\}$ are linearly independent vectors in $\mathbb{R}^n$. Set S is known as the *sampling grid*. Let $C \subset \mathbb{R}^n$ be a bounded set such that $\mathbf{0} \in C$, $C \cap S = \{\mathbf{0}\}$ and

$$S \oplus C = \mathbb{R}^n, \tag{4.52}$$

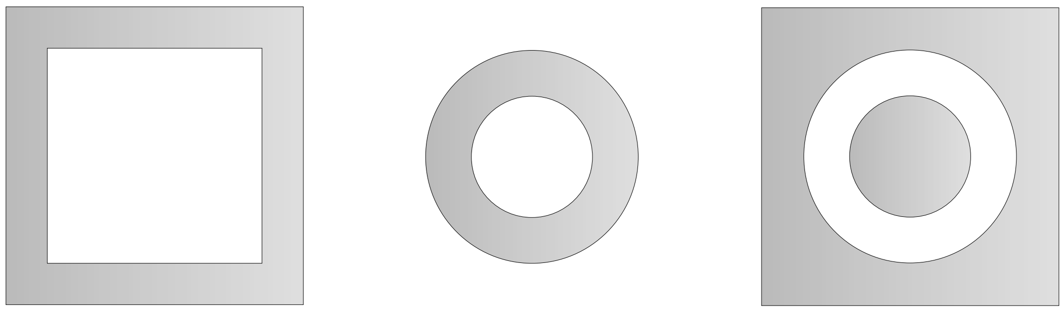


Figure 4.16: Example of a structural closing that is not connectivity-preserving. Note that A and B are connected, but $A \bullet B$ is not. Topological connectivity on $\mathcal{P}(\mathbb{R}^2)$ is assumed.

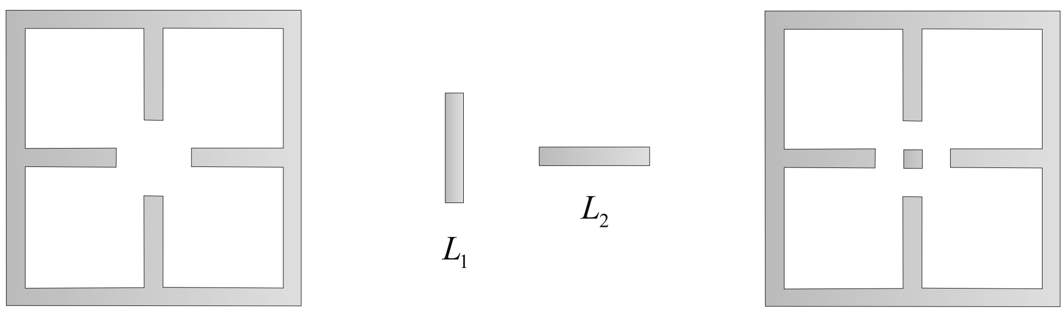


Figure 4.17: A subset $A \in \mathcal{P}(\mathbb{R}^2)$ and its closing $\phi(A) = (A \bullet L_1) \cap (A \bullet L_2)$. Note that A is connected, but $\phi(A)$ is not.

where $\mathbf{0}$ is the origin in $\mathbb{R}^n$. The set C is known as the *sampling element*. One can show that the operator $\sigma(A) = \{s \in S \mid C_s \cap A \neq \emptyset\}$, known as the *sampling operator*, defines a dilation from $\mathcal{P}(\mathbb{R}^n)$ into $\mathcal{P}(S)$. The corresponding adjoint erosion from $\mathcal{P}(S)$ into $\mathcal{P}(\mathbb{R}^n)$, called the *interpolation operator*, is given by $\xi(V) = \left(\bigcup_{s \in S \setminus V} C_s\right)^c$. The sampling operator followed by the interpolation operator produces an operator on $\mathcal{P}(\mathbb{R}^n)$, given by

$$\pi(A) = \xi\sigma(A) = \left(\bigcup_{s \in S} \{C_s \mid C_s \cap A = \emptyset\}\right)^c, \tag{4.53}$$

which is called the *approximation closing*.

In practice, one takes the sampling element to be of the form:

$$C = \{x_1 u_1 + \cdots + x_n u_n \mid -a < x_i < a\}, \quad 1/2 < a \leq 1. \tag{4.54}$$

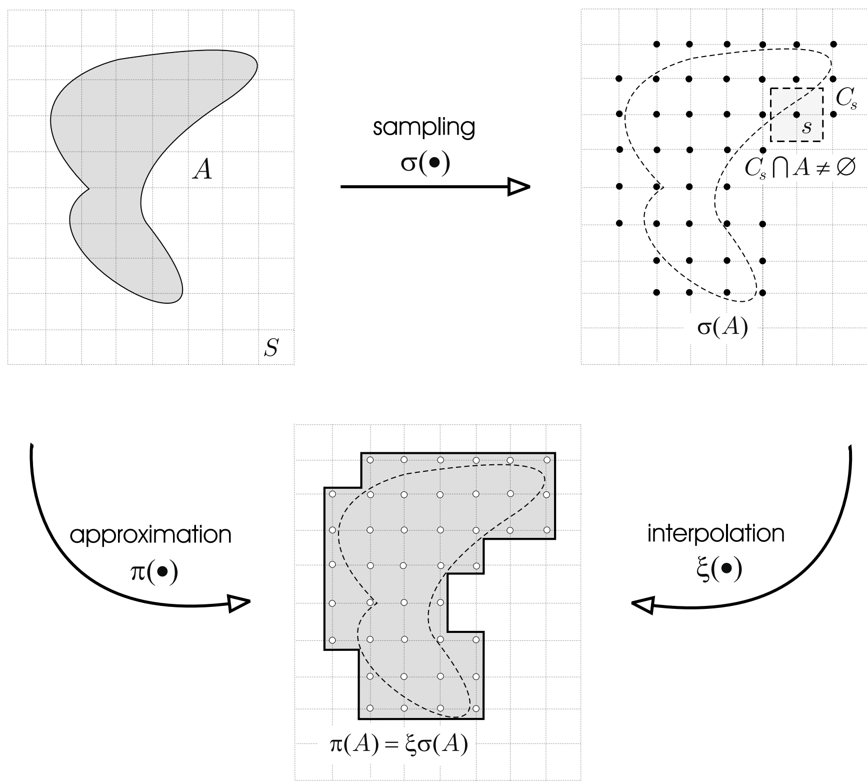


Figure 4.18: Morphological sampling and interpolation.

Fig. 4.18 illustrates this morphological sampling scheme, with the sampling element being chosen as in (4.54).

The following result gives a sufficient condition for the approximation closing π to be a clustering.

4.3.11 Proposition. Consider the lattice $\mathcal{P}(\mathbb{R}^n)$, furnished with a connectivity class $\mathcal{C}$. Let C be a sampling element such that

$$C_s \setminus (R \oplus C) \in \mathcal{C}, \quad \text{for all } s \in S,\ R \subseteq S. \tag{4.55}$$

Then, the approximation closing π, given by (4.53), defines a clustering on $\mathcal{P}(\mathbb{R}^n)$. □

Proof. From Proposition 4.3.8, it suffices to show that π is connectivity-preserving. From (4.52) and (4.53), we have that $\pi(\emptyset) = \left(\bigcup_{s\in S}\{C_s \mid C_s \cap \emptyset = \emptyset\}\right)^c = \left(\bigcup_{s\in S} C_s\right)^c = (S \oplus C)^c = (\mathbb{R}^n)^c = \emptyset$. Hence, $\pi(\emptyset) \in \mathcal{C}$. Let us now consider an element $A \in \mathcal{C} \setminus \{\emptyset\}$. From the extensivity of π, we have $\pi(A) \supseteq A \neq \emptyset$, so that we can pick a point $v \in \pi(A)$. Now, (4.52) guarantees that we can find a point $s(v) \in S$, which depends on v, such that $v \in C_{s(v)}$. Note also that, since $v \in \pi(A)$, we have that $C_{s(v)} \cap A \neq \emptyset$; otherwise, $C_{s(v)} \cap \pi(A) = \emptyset$, as it is clear from (4.53), in which case $v \notin \pi(A)$. Define

$$D(v) = C_{s(v)} \setminus (\pi(A))^c \;=\; C_{s(v)} \setminus (R \oplus C), \tag{4.56}$$

where $R = \{s \in S \mid C_s \cap A = \emptyset\}$. From condition (4.55), we have that $D(v) \in \mathcal{C}$. The following are some additional (and easy to prove) facts about $D(v)$:

$$v \in D(v), \tag{4.57}$$

$$D(v) \subseteq \pi(A), \tag{4.58}$$

$$A \cap D(v) = A \cap C_{s(v)} \neq \emptyset \;\Rightarrow\; A \cup D(v) \in \mathcal{C}. \tag{4.59}$$

From (4.57), it follows that $\pi(A) = \bigcup_{v \in \pi(A)} \{v\} \subseteq \bigcup_{v \in \pi(A)} D(v)$. On the other hand, (4.58) implies that $\bigcup_{v \in \pi(A)} D(v) \subseteq \pi(A)$. Hence, $\pi(A) = \bigcup_{v \in \pi(A)} D(v)$, so that $\pi(A) = A \cup \pi(A) = A \cup \bigcup_{v \in \pi(A)} D(v) = \bigcup_{v \in \pi(A)} (A \cup D(v))$. But $\bigcap_{v \in \pi(A)} (A \cup D(v)) \supseteq A \neq \emptyset$. From (4.59) and axiom (iii) of connectivity classes, this implies that $\pi(A) \in \mathcal{C}$. Q.E.D.

Condition (4.55) is easy to check in practice. In addition, if the connectivity is translation-invariant, then (4.55) can be replaced by $C \setminus R \oplus C \in \mathcal{C}$, for every $R \subseteq S$. In practice, however, we must consider only small finite subsets $R \subseteq S$, which consist of points r for which $C \cap C_r \neq \emptyset$. This simplifies matters even further. For instance, in the case of the sampling element given by (4.54), one needs to consider only combinations of the nearest neighbors to the origin, and some cases are redundant, due to symmetry. In particular, it is very easy to check that the sampling element in (4.54) satisfies condition (4.55) when $\mathcal{C}$ corresponds to the usual topological connectivity.

We refer to $\mathcal{C}^\pi = \{A \in \mathcal{P}(\mathbb{R}^n) \mid \pi(A) \in \mathcal{C}\}$ as the *sampling-based connectivity class* generated by π. Fig. 4.19 illustrates an example of this connectivity class in the 2-D Euclidean space. It is easy to see that condition (4.55) is satisfied in this case. Although A is not connected in the usual topological sense, it is connected in $\mathcal{C}^\pi$, since $\pi(A)$ is connected in the usual topological sense. Clearly, the coarser the sampling grid and the sampling element are, the more subsets of $\mathbb{R}^2$ are connected, in the sense of the associated sampling-based connectivity class.

As shown by Example 4.3.10(a), there exist connectivity-preserving closings that create connected components, and therefore are not strong clusterings. The question thus arises

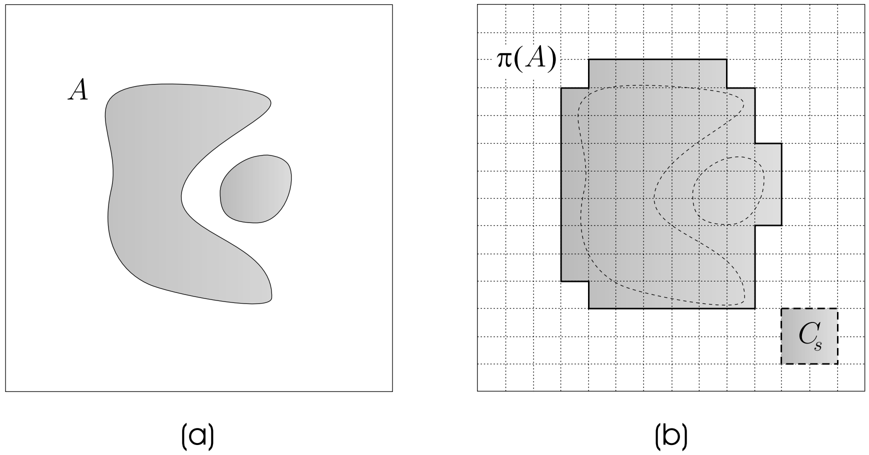


Figure 4.19: An example of a sampling-based connectivity class $\mathcal{C}^\pi$. (a) A subset A of the 2-D Euclidean space that is not connected in the usual topological sense. (b) However, A is connected in $\mathcal{C}^\pi$, since $\pi(A)$ is connected in the usual topological sense.

as to which connectivity-preserving closings ϕ are strong clusterings. It turns out that such closings can be characterized by means of the operators $\xi_x = \gamma_x\phi$, for $x \in \mathcal{S}$. To show this, we first need the following lemma.

4.3.12 Lemma. Let $\mathcal{L}$ be a lattice with a sup-generating family $\mathcal{S}$, furnished with a connectivity class $\mathcal{C}$, and let ϕ be a closing on $\mathcal{L}$ such that $\phi(O) = O$. Then, ϕ is connectivity-preserving if and only if the connected components of $\phi(A)$ are closed, for every $A \in \mathcal{L}$; i.e.,

$$\phi(\mathcal{C}) \subseteq \mathcal{C} \quad \Leftrightarrow \quad \phi\gamma_x\phi = \gamma_x\phi, \ \ x \in \mathcal{S}. \tag{4.60}$$

□

Proof. "⇒": Let $A \in \mathcal{L}$. If $x \not\leq \phi(A)$, then the right-hand side follows from the fact that $\phi(O) = O$. So, let $x \leq \phi(A)$. By the extensivity of ϕ, we have that $\phi\gamma_x\phi(A) \geq \gamma_x\phi(A)$. On the other hand, $\gamma_x\phi(A) \in \mathcal{C} \Rightarrow \phi\gamma_x\phi(A) \in \mathcal{C}$, by hypothesis, and $x \leq \gamma_x\phi(A) \Rightarrow x \leq \phi\gamma_x\phi(A)$. This implies that $\phi\gamma_x\phi(A) \leq \gamma_x\phi(A)$. Hence, $\phi\gamma_x\phi(A) = \gamma_x\phi(A)$.

"⇐": We have that $\phi\gamma_x\phi = \gamma_x\phi \Rightarrow \phi\gamma_x\phi\gamma_x = \gamma_x\phi\gamma_x$. But $\phi\gamma_x\phi\gamma_x \leq \phi\phi\phi\gamma_x = \phi\gamma_x$, and $\phi\gamma_x\phi\gamma_x \geq \phi\gamma_x\gamma_x\gamma_x = \phi\gamma_x$, so that $\phi\gamma_x$ is idempotent; i.e., $\phi\gamma_x\phi\gamma_x = \phi\gamma_x$. Therefore, $\gamma_x\phi\gamma_x = \phi\gamma_x$, for $x \in \mathcal{S}$. Now, we have that $\phi(O) = O \in \mathcal{C}$. So, let $A \in \mathcal{C} \setminus \{O\}$, and pick a sup-generator $x \leq A$. We have that $\gamma_x(A) = A$, so $\gamma_x\phi(A) = \gamma_x\phi\gamma_x(A) = \phi\gamma_x(A) = \phi(A)$; i.e., $\phi(A) \in \mathcal{C}$. Q.E.D.

4.3.13 Proposition. Let $\mathcal{L}$ be a lattice with sup-generating family $\mathcal{S}$, furnished with a connectivity class $\mathcal{C}$, and let ϕ be a closing on $\mathcal{L}$ such that ϕ is connectivity-preserving and $\phi(O) = O$. Then, ϕ is a strong clustering if and only if the operators $\xi_x = \gamma_x \phi$ are strong filters, for all $x \in \mathcal{S}$. □

PROOF. "⇒": From Proposition 2.2.3, ξ_x is a sup-filter, for all $x \in \mathcal{S}$. We need to show that ξ_x is an inf-filter as well; i.e., we need to show that $\gamma_x \phi = \gamma_x \phi(\mathbf{id} \wedge \gamma_x \phi)$, for all $x \in \mathcal{S}$. But this follows by applying γ_x on both sides of (4.44).

"⇐": Since ξ_x is an inf-filter, we have that $\gamma_x \phi = \gamma_x \phi(\mathbf{id} \wedge \gamma_x \phi) \leq \phi(\mathbf{id} \wedge \gamma_x \phi)$. On the other hand, from Lemma 4.3.12, we have that $\phi(\mathbf{id} \wedge \gamma_x \phi) \leq \phi \gamma_x \phi = \gamma_x \phi$. Hence, $\gamma_x \phi = \phi(\mathbf{id} \wedge \gamma_x \phi)$, for all $x \in \mathcal{S}$, which shows (4.44). Hence, ϕ is a strong clustering. Q.E.D.

A class of connectivity-preserving closings, known as partition closings, was introduced by G. Matheron and J. Serra in [77] for the binary case, and extended to the general lattice case by J. Serra in [78]. The following definition is adapted from [78].

4.3.14 Definition. Let $\mathcal{L}$ be a lattice with sup-generating family $\mathcal{S}$, furnished with a connectivity class $\mathcal{C}$. A *partition closing* ϕ is a closing on $\mathcal{L}$ such that $\phi(O) = O$, ϕ is connectivity-preserving, and

$$\phi = \bigvee_{x \in \mathcal{S}} \phi(\mathbf{id} \wedge \gamma_x \phi). \tag{4.61}$$

△

We have the following result.

4.3.15 Proposition. Let $\mathcal{L}$ be a strongly semi-atomic lattice. A closing ϕ on $\mathcal{L}$ is a strong clustering if and only if ϕ is a partition closing. □

PROOF. "⇐": From Definition 4.3.14, it is clear that ϕ is increasing and extensive, and $\phi(O) = O$. We need to show (4.44). Since ϕ is connectivity-preserving, we have that $\phi(\mathbf{id} \wedge \gamma_x \phi) \leq \phi \gamma_x \phi = \gamma_x \phi$, by Lemma 4.3.12. To show the reverse inequality, let $A \in \mathcal{L}$ and $x \in \mathcal{S}$. If $x \not\leq \phi(A)$, the desired result follows from the fact that $\phi(O) = O$. So, let $x \leq \phi(A)$ and take y to be any sup-generator such that $y \leq \gamma_x \phi(A)$. Note that $\gamma_y \phi(A) = \gamma_x \phi(A)$. From (4.61), we have that $y \leq \phi(A) = \bigvee_{x' \in \mathcal{S}} \phi(A \wedge \gamma_{x'} \phi(A))$. From the strong semi-atomicity of $\mathcal{L}$, there must be an $x' \in \mathcal{S}$ such that $y \leq \phi(A \wedge \gamma_{x'} \phi(A)) \leq \gamma_{x'} \phi(A)$. This implies that $\gamma_{x'} \phi(A) = \gamma_y \phi(A) = \gamma_x \phi(A)$. Hence, $y \leq \phi(A \wedge \gamma_x \phi(A))$, for all $y \leq \gamma_x \phi(A)$, so that $\gamma_x\phi(A) = \bigvee\{y \mid y \leq \gamma_x\phi(A)\} \leq \phi(A \wedge \gamma_x\phi(A))$. Applying γ_x on both sides, we get that $\gamma_x\phi(A) \leq \gamma_x\phi(A \wedge \gamma_x\phi(A))$. Hence, $\gamma_x\phi = \gamma_x\phi(\mathbf{id} \wedge \gamma_x\phi)$, for all $x \in \mathcal{S}$, as required.

"$\Rightarrow$": By definition, $\phi(O) = O$ and, as we showed in Proposition 4.3.3, ϕ is connectivity-preserving. In addition, from (4.44), we have that $\phi = \bigvee_{x \in \mathcal{S}} \gamma_x\phi = \bigvee_{x \in \mathcal{S}} \phi(\mathbf{id} \wedge \gamma_x\phi)$, which shows (4.61). Hence, ϕ is a partition closing. Q.E.D.

We remark that the argument used above for the proof of the converse implication is a generalization, to the case of a strongly semi-atomic lattice, of the argument used in the proof of Proposition 7.8 in [77]. In that reference, iterative algorithms are given that generate a partition closing from a given connectivity-preserving closing, in the case of the binary lattice $\mathcal{L} = \mathcal{P}(E)$, with finite E. Since this lattice is strongly semi-atomic, this provides a general method to "strengthen" a given discrete binary connectivity-preserving closing to obtain a strong clustering.

4.3.2 Connectivity Based on Contraction

A *contraction* ξ is any increasing and anti-extensive operator on a lattice $\mathcal{L}$ (this follows the terminology adopted, for different purposes, in [65]). We say that $A \in \mathcal{L}$ is "stable" if $\xi(A) = A$; i.e., if A is invariant to the contraction ξ. A new connectivity class can be generated from a given connectivity class by means of a contraction. The resulting connectivity class contains the least element, the sup-generators, and the stable connected elements, and is thus a restricted version of the given connectivity class. Some of the results presented in this subsection are based on the work by C. Ronse [65, 66]. However, we extend Ronse's results from binary lattices to general atomic lattices (and to more general lattices if possible), and provide new examples that are made possible by this extension.

We have the following result.

4.3.16 Proposition. Consider an atomic lattice $\mathcal{L}$ with sup-generating family $\mathcal{S}$, furnished with a connectivity class $\mathcal{C}$. Let ξ be a contraction on $\mathcal{L}$. The family

$$\mathcal{C}^{\xi} = \{O\} \cup \mathcal{S} \cup \{A \in \mathcal{C} \mid \xi(A) = A\} \tag{4.62}$$

is a connectivity class in $\mathcal{L}$ that is smaller than $\mathcal{C}$ (i.e., $\mathcal{C}^{\xi} \subseteq \mathcal{C}$). □

Proof. From (4.62), it is clear that $\mathcal{C}^{\xi} \subseteq \mathcal{C}$. In addition, $\mathcal{C}^{\xi}$ satisfies axioms (*i*) and (*ii*) of connectivity classes. We now show axiom (*iii*). Consider a family $\{C_\alpha \mid \alpha \in J\}$ in $\mathcal{C}^{\xi}$.

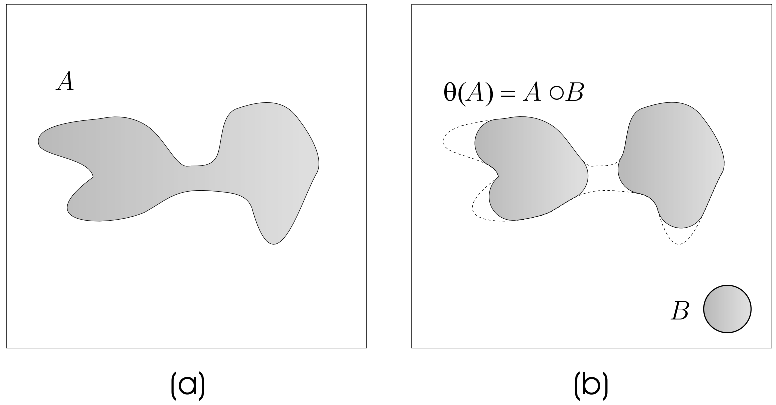


Figure 4.20: An example of an opening-based connectivity class $\mathcal{C}^\theta$. (a) A set A in the 2-D Euclidean space that is connected in the usual topological sense. (b) The set A is not connected in $\mathcal{C}^\theta$, since $A \neq A \circ B$.

Let $\mathcal{D} = \{A \in \mathcal{C} \mid \xi(A) = A\}$. Since $O \in \mathcal{D}$, we can write $C_\alpha \in \mathcal{S}$, for $\alpha \in J_1$, and $C_\alpha \in \mathcal{D} \setminus \mathcal{S}$, for $\alpha \in J \setminus J_1$, where $J_1 \subseteq J$. Note that, since the elements of $\mathcal{S}$ are atoms, we have

$$x \wedge A \neq O \;\Rightarrow\; x \wedge A = x \;\Rightarrow\; A = x \text{ or } x \leq A \notin \mathcal{S}, \tag{4.63}$$

for all $x \in \mathcal{S}$ and $A \in \mathcal{L}$. Now, suppose that $\bigwedge_{\alpha \in J} C_\alpha \neq O$. It follows from (4.63) that $C_\alpha = x$, for $\alpha \in J_1$, where x is a fixed element of $\mathcal{S}$. If $J_1 = J$, we have that $\bigvee_{\alpha \in J} C_\alpha = x \in \mathcal{C}^\xi$, and we are done. Otherwise, it follows from (4.63) that $\bigvee_{\alpha \in J} C_\alpha = \bigvee_{\alpha \in J \setminus J_1} C_\alpha$. But, for all $\alpha \in J \setminus J_1$, we have $\xi(C_\alpha) = C_\alpha$, so that $\bigvee_{\alpha \in J \setminus J_1} C_\alpha = \bigvee_{\alpha \in J \setminus J_1} \xi(C_\alpha) \leq \xi(\bigvee_{\alpha \in J \setminus J_1} C_\alpha) \Rightarrow \bigvee_{\alpha \in J \setminus J_1} C_\alpha = \xi(\bigvee_{\alpha \in J \setminus J_1} C_\alpha)$, by using the fact that ξ is increasing and anti-extensive. We also have $C_\alpha \in \mathcal{C}$, for all $\alpha \in J \setminus J_1$, with $\bigwedge_{\alpha \in J \setminus J_1} C_\alpha \geq \bigwedge_{\alpha \in J} C_\alpha \neq O$, so that $\bigvee_{\alpha \in J \setminus J_1} C_\alpha \in \mathcal{C}$, since $\mathcal{C}$ is a connectivity class. It follows that $\bigvee_{\alpha \in J} C_\alpha = \bigvee_{\alpha \in J \setminus J_1} C_\alpha \in \mathcal{D} \Rightarrow \bigvee_{\alpha \in J} C_\alpha \in \mathcal{C}^\xi$, as required. Q.E.D.

The connectivity class $\mathcal{C}^\xi$ is referred to as a *contraction-based connectivity class*. A contraction-based connectivity class $\mathcal{C}^\xi$ restricts the original connectivity class $\mathcal{C}$ by excluding all connected elements that are not stable, according to ξ.

In the case in which ξ is an opening $\xi = \theta$ on $\mathcal{L}$, the connectivity class $\mathcal{C}^\theta$ defined by (4.62), with $\xi = \theta$, is referred to as the *opening-based connectivity class* generated by θ. In the special case in which $\mathcal{L} = \mathcal{P}(E)$, with the points as sup-generators, this connectivity class is the one discussed in [65, 66]. Fig. 4.20 illustrates this case. In this example, $\mathcal{L} = \mathcal{P}(\mathbb{R}^2)$, the sup-generators are the points in $\mathbb{R}^2$, and θ is taken to be the structural opening $\theta(A) = A \circ B$, with B being a two-dimensional disk. Although the set A is connected in the usual topological sense, it is not connected in $\mathcal{C}^\theta$, since A is not invariant to θ. Therefore, one-piece objects with "bottlenecks" are not considered connected in the new connectivity class (this affords robustness against noise). Clearly, the larger the radius of B is, the fewer subsets of $\mathbb{R}^2$ are connected in $\mathcal{C}^\theta$.

Under additional assumptions, we can characterize the connectivity openings of the opening-based connectivity class $\mathcal{C}^\theta$ defined by (4.62), with $\xi = \theta$. First, we need the following definition.

4.3.17 Definition. Consider a lattice $\mathcal{L}$ with sup-generating family $\mathcal{S}$, furnished with a connectivity class $\mathcal{C}$. An opening θ on $\mathcal{L}$ is said to be *locally invariant* with respect to $\mathcal{C}$ if, for each $A \in \mathcal{L}$,

$$\theta(A) = A \;\Rightarrow\; \theta\gamma_x(A) = \gamma_x(A), \quad \forall x \in \mathcal{S}. \tag{4.64}$$

△

In other words, given an element A that is invariant to θ, all connected components of A must also be invariant to θ. This can also be expressed by writing $\theta\gamma_x\theta = \gamma_x\theta$, for all $x \in \mathcal{S}$. We now have the following characterization of locally invariant openings on infinite $\vee$-distributive lattices.

4.3.18 Proposition. Consider an infinite $\vee$-distributive lattice $\mathcal{L}$, furnished with a connectivity class $\mathcal{C}$. An opening θ on $\mathcal{L}$ is locally invariant with respect to $\mathcal{C}$ if and only if there exists a family $\mathcal{B} \subseteq \mathcal{C}$ such that

$$\theta(A) = \theta_{\mathcal{B}}(A) = \bigvee\{B \in \mathcal{B} \mid B \leq A\}. \tag{4.65}$$

□

Proof. "$\Rightarrow$": Let θ be locally invariant with respect to $\mathcal{C}$; we show that $\theta = \theta_{\mathcal{B}}$, with $\mathcal{B} = \mathcal{C} \cap \mathrm{Inv}(\theta)$. Let $C = \gamma_x\theta(A)$, where $x \leq \theta(A)$. Since $\theta(A) \in \mathrm{Inv}(\theta)$ and θ is locally invariant, we have that $C \in \mathrm{Inv}(\theta)$. But, we also have $C \in \mathcal{C}$, so that $C \in \mathcal{C} \cap \mathrm{Inv}(\theta) = \mathcal{B}$. Since $C \leq A$ and $\theta_{\mathcal{B}}(A) = \bigvee\{B \in \mathcal{B} \mid B \leq A\}$, we have $x \leq C \leq \theta_{\mathcal{B}}(A)$, so that $\theta(A) = \bigvee_{x \leq \theta(A)} x \leq \theta_{\mathcal{B}}(A)$. To show the reverse inequality, let $B \in \mathcal{B}$ such that $B \leq A$. We have $B = \theta(B) \leq \theta(A)$, so that $\theta_{\mathcal{B}}(A) \leq \theta(A)$. Hence, $\theta(A) = \theta_{\mathcal{B}}(A)$.

"$\Leftarrow$": Let $\theta = \theta_{\mathcal{B}}$, for some $\mathcal{B} \subseteq \mathcal{C}$. We show that θ is locally invariant with respect to $\mathcal{C}$; i.e., if $A = \theta_{\mathcal{B}}(A)$ and $C \lessdot A$, then $C = \theta_{\mathcal{B}}(C)$. Let $\mathcal{B}(C) = \{B \in \mathcal{B} \mid B \leq A,\, C \wedge B \neq O\}$.

We have that

$$C \leq A \Rightarrow C = C \wedge A = C \wedge \bigvee \{B \mid B \in \mathcal{B},\, B \leq A\}$$
$$= \bigvee \{C \wedge B \mid B \in \mathcal{B},\, B \leq A\}$$
$$= \bigvee \{C \wedge B \mid B \in \mathcal{B}(C)\}, \tag{4.66}$$

where we have used the infinite $\vee$-distributivity of $\mathcal{L}$. Now, let $B \in \mathcal{B}(C)$. This implies that $B \in \mathcal{C}$, with $C \wedge B \neq O$. But $C \in \mathcal{C}$, so that $C \vee B \in \mathcal{C}$. But, we also have that $C \vee B \leq A$ and $C \lessdot A$; hence, $C \vee B = C \Rightarrow C \geq B \Rightarrow C \wedge B = B$. From (4.66), we can thus write

$$C = \bigvee \{B \mid B \in \mathcal{B}(C)\} \leq \bigvee \{B \in \mathcal{B} \mid B \leq C\} = \theta_{\mathcal{B}}(C). \tag{4.67}$$

But, of course, we also have that $\theta_{\mathcal{B}}(C) \leq C$, which establishes the desired equality. Q.E.D.

Proposition 4.3.18 extends the corresponding binary result in [65] to infinite $\vee$-distributive lattices. Note that the infinite $\vee$-distributivity of $\mathcal{L}$ is only required for the converse implication. The following is a useful corollary.

4.3.19 Corollary. Let $\mathcal{L}$ be an infinite $\vee$-distributive lattice, with sup-generating family $\mathcal{S}$, furnished with a connectivity class $\mathcal{C}$. Let δ be a dilation on $\mathcal{L}$ such that $\delta(x) \in \mathcal{C}$, for all $x \in \mathcal{S}$, and let ϵ be the adjunct erosion on $\mathcal{L}$. Then, the adjunctional opening $\theta = \delta\epsilon$ is locally invariant with respect to $\mathcal{C}$. □

Proof. From the fact that (ϵ, δ) is an adjunction, we have that $\epsilon(A) = \bigvee_{x \in \mathcal{S}} \{x \mid x \leq \epsilon(A)\} = \bigvee_{x \in \mathcal{S}} \{x \mid \delta(x) \leq A\}$. Hence, $\theta(A) = \delta\epsilon(A) = \delta(\bigvee_{x \in \mathcal{S}} \{x \mid \delta(x) \leq A\}) = \bigvee_{x \in \mathcal{S}} \{\delta(x) \mid \delta(x) \leq A\} = \theta_{\mathcal{B}}(A)$, with $\mathcal{B} = \{\delta(x) \mid x \in \mathcal{S}\} \subseteq \mathcal{C}$. The desired result then follows from Proposition 4.3.18. Q.E.D.

For instance, consider a translation-invariant connectivity class $\mathcal{C}$ in $\mathcal{L} = \mathcal{P}(\mathbb{R}^n)$. If B is a connected structuring element, then, from Corollary 4.3.19, it is clear that the structural opening $\theta_B(A) = A \circ B$ is locally invariant with respect to $\mathcal{C}$. The assumption that B is connected is essential. For instance, consider the Euclidean topological connectivity on $\mathbb{R}^n$, and take $B = \{v_1, v_2\}$, where v_1, v_2 are two distinct points in $\mathbb{R}^n$. Let $A = R_1 \cup R_2$, where R_1 and R_2 are parallel lines going through points v_1 and v_2, respectively. Then, $\theta_B(A) = A$, so that A is invariant to θ_B, but $\theta_B(R_1) = \theta_B(R_2) = \emptyset$; i.e., the connected components R_1 and R_2 of A are not invariant to θ_B.

Proposition 4.3.18 and Corollary 4.3.19 provide ways for building openings that are locally invariant with respect to a given connectivity class. The usefulness of such openings becomes clear from the following result.

4.3.20 Proposition. Consider an atomic lattice $\mathcal{L}$ with sup-generating family $\mathcal{S}$, furnished with a connectivity class $\mathcal{C}$. Let $\{\gamma_x \mid x \in \mathcal{S}\}$ and ρ be the connectivity openings and the reconstruction operator, respectively, associated with $\mathcal{C}$. Let θ be an opening on $\mathcal{L}$ that is locally invariant with respect to $\mathcal{C}$, and let $\mathcal{C}^\theta$ be the opening-based connectivity class generated by θ.

(a) The connectivity openings $\{\gamma_x^\theta \mid x \in \mathcal{S}\}$ associated with $\mathcal{C}^\theta$ are given by

$$\gamma_x^\theta(A) = \begin{cases} \gamma_x\theta(A), & \text{if} \quad x \leq \theta(A) \\ x, & \text{if} \quad \theta(A) \ngeq x \leq A\,, \\ O, & \text{if} \quad x \nleq A \end{cases} \quad \text{for } x \in \mathcal{S}, \tag{4.68}$$

for $A \in \mathcal{L}$.

(b) The reconstruction operator ρ^θ associated with $\mathcal{C}^\theta$ is given by

$$\rho^\theta(A \mid M) = (A \wedge M) \vee \rho(\theta(A) \mid M), \tag{4.69}$$

for $A, M \in \mathcal{L}$. □

Proof. (a): From the definition of connectivity openings, we have that $\gamma_x^\theta(A) = \bigvee\{C \in \mathcal{C}^\theta \mid x \leq C \leq A\}$. Let us consider three cases. If $x \nleq A$, then clearly $\gamma_x^\theta(A) = O$. If $x \leq A$, but $x \nleq \theta(A)$, consider $C \in \mathcal{C}^\theta$ such that $x \leq C \leq A$. We have $\theta(C) \leq \theta(A)$, so that we cannot have $C = \theta(C)$; i.e., $C \notin \mathcal{C} \cap \mathrm{Inv}(\theta)$. Hence, C must be a sup-generator. By the atomicity of $\mathcal{L}$, we must have $C = x$, so that $\gamma_x^\theta(A) = x$. Finally, if $x \leq \theta(A)$, we have that $x \leq \gamma_x\theta(A) \leq A$. But $\gamma_x\theta(A) \in \mathcal{C}^\theta$, since $\gamma_x\theta(A) \in \mathcal{C}$, and $\gamma_x\theta(A) \in \mathrm{Inv}(\theta)$, by the local-invariance of θ with respect to $\mathcal{C}$. Hence, we have that $\gamma_x\theta(A) \leq \gamma_x^\theta(A)$. To show the reverse inequality, we consider only the case when $\gamma_x^\theta(A) \in \mathcal{C} \cap \mathrm{Inv}(\theta)$ (if $\gamma_x^\theta(A)$ is a sup-generator, then the direct inequality clearly implies that $\gamma_x\theta(A) = \gamma_x^\theta(A) = x$). From $\gamma_x^\theta(A) \in \mathcal{C}$, we have $\gamma_x^\theta(A) = \gamma_x\gamma_x^\theta(A)$, whereas, from $\gamma_x^\theta(A) \in \mathrm{Inv}(\theta)$, we have $\gamma_x^\theta(A) = \theta\gamma_x^\theta(A)$. It follows that $\gamma_x^\theta(A) = \gamma_x\theta\gamma_x^\theta(A) \leq \gamma_x\theta(A)$, since γ_x is anti-extensive and $\gamma_x\theta$ is increasing.

(b): The result follows from part (a) and the fact that $\rho^\theta(A \mid M) = \bigvee_{x \leq M} \gamma_x^\theta(A)$. Q.E.D.

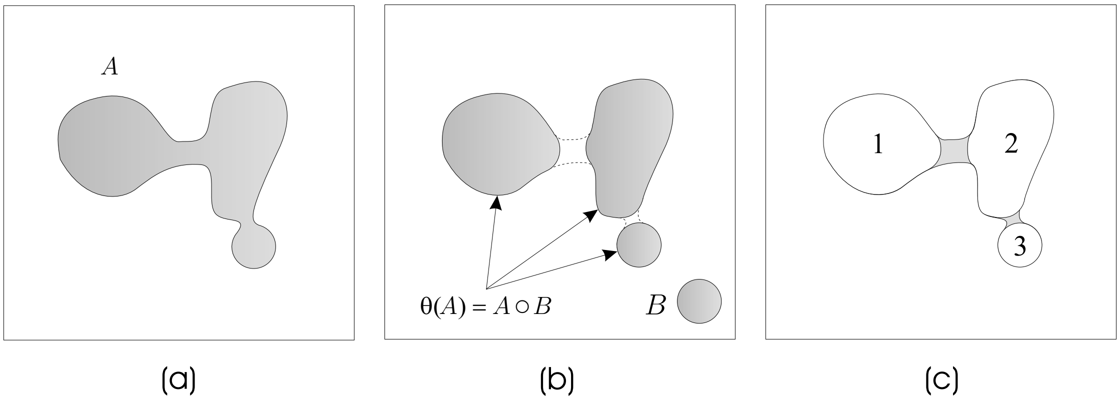


Figure 4.21: (a) A set A in the 2-D Euclidean space. (b) The structural opening $\theta(A) = A \circ B$. (c) The PCC of A according to the opening-based connectivity class generated by θ. The gray regions correspond to the residue $A \setminus \theta(A)$, where the PCC consists of a pulverization into points.

Note that, when $M = x$, (4.69) reduces to (4.68), as would be expected in an atomic lattice (see remarks following (4.18)). Note also that, in the standard binary case (i.e., when $\mathcal{L} = \mathcal{P}(E)$ with the points in E as sup-generators), (4.68) can be written as

$$\gamma^{\theta}_{\{v\}}(A) = \begin{cases} \gamma_{\{v\}}\theta(A), & \text{if} \quad v \in \theta(A) \\ \{v\}, & \text{if} \quad v \in A \setminus \theta(A) \, . \\ \emptyset, & \text{if} \quad v \notin A \end{cases} \tag{4.70}$$

Equation (4.68) and its binary counterpart (4.70) simply say that the connected component, according to the opening-based connectivity class generated by an opening θ, of an element A marked by x, is the connected component, according to the original connectivity class, of $\theta(A)$, marked by x, if $x \leq \theta(A)$, or x itself, if $x \leq A$ but $x \not\leq \theta(A)$. This "breaks up" connected elements in $\mathcal{C}$ by means of the opening θ. An example is depicted in Fig. 4.21. Here, we consider the 2-D Euclidean case, with θ being the structural opening $\theta(A) = A \circ B$, where B is a two-dimensional disk. According to Corollary 4.3.19, this opening is locally invariant.

The requirement that θ be locally invariant is not needed for generating an opening-based connectivity class (see Proposition 4.3.16). This requirement is only needed in order to express the connectivity opening γ^{θ}_x in terms of the connectivity opening γ_x, by means of (4.68). As a matter of fact, if θ is not locally invariant, it can be shown that the opening-based connectivity class generated by θ is identical to the one generated by the locally

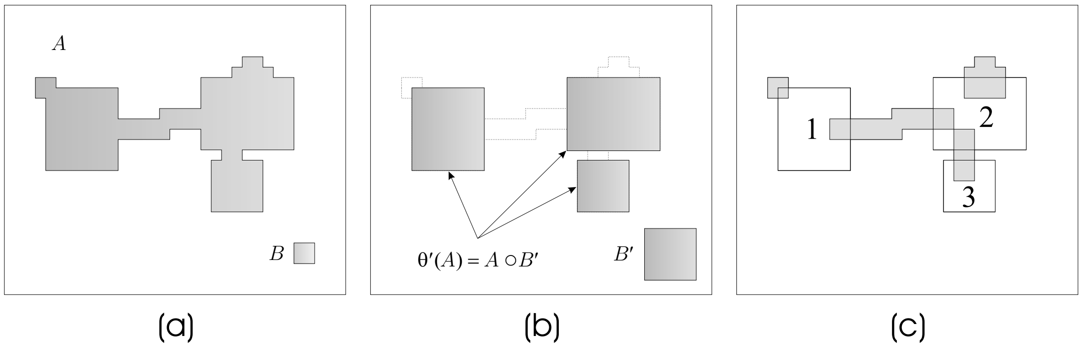


Figure 4.22: (a) A B-open image A in $\mathcal{L}^\theta$, where B is a square structuring element. (b) The structural opening $\theta'(A) = A \circ B'$, where B' is a square structuring element that is larger than B. (c) The PCC of A according to the opening-based connectivity class $\mathcal{C}^{\theta'}$. The gray regions depict the union of the sup-generators of $\mathcal{L}^\theta$ that are in A, but not in $\theta'(A)$.

invariant opening $\theta_{\mathcal{B}}$, given by (4.65), with $\mathcal{B} = \mathcal{C} \cap \mathrm{Inv}(\theta)$. In this case, (4.68) holds with θ being replaced by $\theta_{\mathcal{B}}$ (the proof of this fact is straightforward).

As an application of the results presented in this section, we give below an example of an opening-based connectivity class in a non-binary atomic lattice.

4.3.21 Example. Consider the θ-invariant lattice $\mathcal{L}^\theta = \mathrm{Inv}(\theta)$, where $\theta(A) = A \circ B$ on $\mathcal{L} = \mathcal{P}(E)$ (see Section 4.2), and let $\mathcal{C}^\theta$ be the connectivity class in $\mathcal{L}^\theta$, given by Proposition 4.2.3 (this requires that $B_v \in \mathcal{C}$, for all $v \in E$). Recall that $\mathcal{L}^\theta$ is sup-generated by the family $\mathcal{S}^\theta = \{B_v \mid v \in E\}$, so that $\mathcal{L}^\theta$ is clearly an atomic lattice. Then, given an opening θ' on $\mathcal{L}^\theta$, the family $\mathcal{C}^{\theta'} = \{\emptyset\} \cup \mathcal{S}^\theta \cup [\mathcal{C}^\theta \cap \mathrm{Inv}(\theta')]$ is the opening-based connectivity class in $\mathcal{L}^\theta$ generated by θ'. In particular, consider a structural opening $\theta'(A) = A \circ B'$, where $B' \in \mathcal{C}$ and B' is B-open, which implies that $\mathrm{Inv}(\theta') \subseteq \mathrm{Inv}(\theta)$ (for a proof of this fact, see [34, Thm. 3.24]). Hence, $\theta'(\mathcal{L}^\theta) = \mathrm{Inv}(\theta') \subseteq \mathrm{Inv}(\theta) = \mathcal{L}^\theta$, so that θ' defines an opening on $\mathcal{L}^\theta$. Furthermore, even though $\mathcal{L}^\theta$ is not infinite $\vee$-distributive (so we cannot use Proposition 4.3.18), it is easy to see that θ' is locally invariant with respect to $\mathcal{C}$. Hence, we can use (4.68) to compute the connectivity openings of the opening-based connectivity class $\mathcal{C}^{\theta'}$. Fig. 4.22 illustrates this, where $\mathcal{L}^\theta$ consists of the B-open subsets of $\mathbb{R}^2$, with B being a square structuring element, $\mathcal{C}^\theta$ corresponds to the B-open topologically connected sets in the Euclidean topology of $\mathbb{R}^2$, and B' is a square structuring element that is larger than B. Note that the zones of the PCC may overlap, but the infimum between each pair of zones is zero in lattice $\mathcal{L}^\theta$, as required. $\diamondsuit$

4.4 Hyperconnectivity

Axiom (*iii*) of connectivity classes requires that the supremum of "overlapping" connected elements (i.e., connected elements that have nonzero infimum) must be connected (see Definition 4.1.2). In some cases, this may be too restrictive. For example, this is true when one tries to define a connectivity class in the function lattice $\mathrm{Fun}(E, \mathcal{T})$.

In Section 4.2, we have seen that there is a way to soften this constraint, by introducing new lattices where the infimum operation is suitably "modified." An alternative approach is the concept of hyperconnectivity, proposed by J. Serra in [78, 79]. The hyperconnectivity approach modifies axiom (*iii*), by introducing an "overlap" operation that generalizes the nonzero infimum criterion. The disadvantage of this approach is that it loses much of the structure and strength of the theory of connectivity classes. The benefits come from the possibility of defining examples of connectivity on usual lattices such as $\mathcal{P}(E)$ or $\mathrm{Fun}(E, \mathcal{T})$, using meaningful overlap criteria, which could not be otherwise achieved if one restricts oneself to connectivity classes. In particular, we show that graph-theoretical k-connectivity, and fuzzy level connectivity and fuzzy topographic connectivity, have natural formulations as hyperconnectivities on $\mathcal{P}(E)$ and $\mathrm{Fun}(E, \mathcal{T})$, respectively.

In this section, we study this alternative approach to connectivity. Most of the material presented here is new, although it builds on the original idea proposed by J. Serra in [78, 79].

4.4.1 Hyperconnectivity Classes

Below, we provide an axiomatization of hyperconnectivity, based on an "overlap" criterion for families in a lattice. Overlap criteria extend the usual criterion of nonzero infimum adopted in the case of connectivity classes.

4.4.1 Definition. An *overlap criterion* in a lattice $\mathcal{L}$ is a mapping $\perp\colon \mathcal{P}(\mathcal{L}) \to \mathcal{K}$, where $\mathcal{K} = \{O_\perp, I_\perp\}$ is a bi-valued chain, such that:

$$\mathcal{A} \subseteq \mathcal{B} \;\Rightarrow\; \perp(\mathcal{A}) \geq \perp(\mathcal{B}). \tag{4.71}$$

A family $\mathcal{A} \subseteq \mathcal{L}$ is said to be *overlapping* if $\perp(\mathcal{A}) = I_\perp$; otherwise, it is said to be *non-overlapping.* △

The condition expressed by (4.71) conveys the natural requirement that a non-overlapping family cannot possibly become overlapping by adding more elements. For notational convenience, we write $A_1 \perp A_2$ if $\perp(\{A_1, A_2\}) = I_\perp$, and $A_1 \not\perp A_2$ if $\perp(\{A_1, A_2\}) = O_\perp$. Recall that definitions of overlapping were already introduced in Chapter 3, in connection of fuzzy level connectivity and fuzzy topographic connectivity; it is clear that those are examples of overlap criteria (see also remarks after Example 4.4.3 below).

We now define hyperconnectivity classes in a lattice $\mathcal{L}$.

4.4.2 Definition. Let $\mathcal{L}$ be a lattice with sup-generating family $\mathcal{S}$, furnished with an overlap criterion $\perp$. A family $\mathcal{H} \subseteq \mathcal{L}$ is called a *hyperconnectivity class* in $\mathcal{L}$ if the following conditions are satisfied:

(*i*) $O \in \mathcal{H}$,

(*ii*) $\mathcal{S} \subseteq \mathcal{H}$,

(*iii*) for a family $\{H_\alpha\}$ in $\mathcal{H}$ such that $\perp(\{H_\alpha\}) = I_\perp$, we have that $\bigvee H_\alpha \in \mathcal{H}$.

The family $\mathcal{H}$ generates a *hyperconnectivity* on $\mathcal{L}$, and the elements of $\mathcal{H}$ are said to be *hyperconnected.* △

It is clear that a connectivity class is a special case of a hyperconnectivity class with the "standard" overlap criterion

$$\perp_\wedge(\mathcal{A}) = \begin{cases} I_\perp, & \text{if } \bigwedge \mathcal{A} \neq O \\ O_\perp, & \text{otherwise} \end{cases}. \tag{4.72}$$

Next, we give a few examples of hyperconnectivity classes.

4.4.3 Example.

(a) Let $\mathcal{L}$ be a lattice with sup-generating family $\mathcal{S}$, furnished with a connectivity class $\mathcal{C}$. Let δ be an extensive dilation on $\mathcal{L}$ such that $\delta(x) \in \mathcal{C}$, for every $x \in \mathcal{S}$, and consider the overlap criterion

$$\perp(\{A_\alpha\}) = \begin{cases} I_\perp, & \text{if } \bigwedge \delta(A_\alpha) \neq O \\ O_\perp, & \text{otherwise} \end{cases}. \tag{4.73}$$

It is easy to verify that the family $\mathcal{H} = \delta^{-1}(\mathcal{C})$ is a hyperconnectivity class in $\mathcal{L}$. This is also the dilation-based connectivity class generated by δ (see Section 4.3.1).

(b) Let $\mathcal{L} = \mathcal{P}(\mathbb{Z}^n)$, with the points as sup-generators, furnished with the overlap criterion

$$\perp(\{A_\alpha\}) = \begin{cases} I_\perp, & \text{if } |\bigcap A_\alpha| \geq k \\ O_\perp, & \text{otherwise} \end{cases}, \tag{4.74}$$

where k is a positive integer. In other words, sets in $\mathbb{Z}^n$ overlap if they have at least k points in common. Let $G = (\mathbb{Z}^n, L)$ be a graph, and recall the definition of graph-theoretic k-connectivity in Section 3.2. It is easy to verify that

$$\mathcal{H} = \{A \subseteq \mathbb{Z}^n \mid A \text{ is a point or } A \text{ is } k\text{-connected in } G\} \tag{4.75}$$

is a hyperconnectivity class in $\mathcal{P}(\mathbb{Z}^n)$.

(c) (*Flat τ-hyperconnectivity*). Let $\mathcal{L} = \mathrm{Fun}(E, \mathcal{T})$ with the pulses as sup-generators. Recall the definition of the threshold operators $Y_\tau(f) = \{v \in E \mid f(v) \not\leq \tau\}$, for $\tau \in \mathcal{T}$. Consider the overlap criterion

$$\perp_\tau(\{f_\alpha\}) = \begin{cases} I_\perp, & \text{if } \bigcap_\alpha \{Y_{\tau'}(f_\alpha) \mid Y_{\tau'}(f_\alpha) \neq \emptyset\} \neq \emptyset, \text{ for all } \tau' \not\geq \tau \\ O_\perp, & \text{otherwise} \end{cases}, \tag{4.76}$$

where $\tau \in \mathcal{T} \setminus \{O\}$. In other words, functions in $\mathrm{Fun}(E, \mathcal{T})$ overlap if all their non-empty threshold sets intersect at all levels "below" level τ. Let $\{\mathcal{C}_\tau \mid \tau \in \mathcal{T} \setminus \{I\}\}$ be a family of connectivity classes in $\mathcal{P}(E)$. For each $\tau \in \mathcal{T} \setminus \{O\}$, it is easy to verify, by using the axioms of connectivity classes and the fact that $Y_t(\bigvee f_\alpha) = \bigcup Y_t(f_\alpha)$, that

$$\mathcal{H}_\tau = \{f \in \mathrm{Fun}(E, \mathcal{T}) \mid Y_{\tau'}(f) \in \mathcal{C}_{\tau'}, \text{ for all } \tau' \not\geq \tau\} \tag{4.77}$$

is a hyperconnectivity class in $\mathrm{Fun}(E, \mathcal{T})$, which is called the *flat τ-hyperconnectivity class* associated with $\{C_\tau\}$. The elements in $\mathcal{H}_\tau$ are called the *flat τ-hyperconnected functions* in $\mathrm{Fun}(E, \mathcal{T})$. A function is flat τ-hyperconnected if all its threshold sets "below" level τ are connected, according to the respective connectivity class. Flat τ-hyperconnectivity defines a degree of connectivity on $\mathrm{Fun}(E, \mathcal{T})$, in the sense that $\mathcal{H}_\tau \subseteq \mathcal{H}_{\tau'}$, for $\tau \geq \tau'$; i.e., flat τ-hyperconnectivity implies flat τ'-hyperconnectivity, for $\tau \geq \tau'$. Flat I-hyperconnectivity is simply referred to as *flat hyperconnectivity*, whereas flat I-hyperconnected functions are simply referred to as *flat hyperconnected functions*. Moreover, we write $\perp$ instead of $\perp_I$ and $\not\perp$ instead of $\not\perp_I$. Clearly, flat hyperconnectivity implies flat τ-hyperconnectivity, for each $\tau \in \mathcal{T} \setminus \{O\}$. $\diamondsuit$

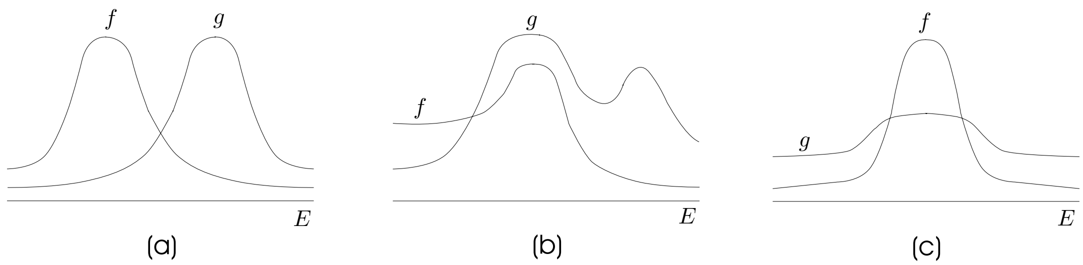


Figure 4.23: An example of flat hyperconnectivity on $\mathrm{Fun}(E, \overline{\mathbb{R}})$, associated with the usual Euclidean topological connectivity on $E = \mathbb{R}$. (a) $f \not\perp g$ and f and g are flat hyperconnected. (b) $f \perp g$ and f is flat hyperconnected, whereas g is not. (c) $f \perp g$ and both f and g are flat hyperconnected. Note that $f \vee g$ is flat hyperconnected only in (c) (in this case, flat hyperconnectivity of the supremum is required by axiom (iii) of hyperconnectivity classes).

Example 4.4.3(a) shows that graph-theoretic k-connectivity has a natural definition as a hyperconnectivity class in the binary lattice $\mathcal{P}(E)$.

The hyperconnectivity class of Example 4.4.3(b) is generated by means of the threshold sets of a function, hence the name "flat" τ-hyperconnectivity. An important special case corresponds to the family $\{\mathcal{C}_\tau = \mathcal{C} \mid \tau \in \mathcal{T} \setminus \{I\}\}$, where $\mathcal{C}$ is some fixed connectivity class in $\mathcal{P}(E)$. In this case, the corresponding flat τ-hyperconnectivity class is said to be associated with $\mathcal{C}$. Fig. 4.23 illustrates such an example of flat hyperconnectivity, which corresponds to the original example of hyperconnectivity proposed by J. Serra in [78, 79] to model the *catchment basins* of the *watershed transform* [6].

Fuzzy level connectivity is an example of flat hyperconnectivity, as shown by the next result.

4.4.4 Proposition. Let (E, Δ) be a $\mathcal{T}$-fuzzy topological space. A function $f \in \mathrm{Fun}(E, \mathcal{T})$ is level connected if and only if it is flat hyperconnected in $\mathrm{Fun}(E, \mathcal{T})$, with respect to the family $\{\mathcal{C}_\tau \mid \tau \in \mathcal{T} \setminus \{I\}\}$ of connectivity classes in $\mathcal{P}(E)$, given by

$$\mathcal{C}_\tau = \{A \in \mathcal{P}(E) \mid A \text{ is connected in the topological space } (E, Y_\tau(\Delta))\}, \tag{4.78}$$

for $\tau \in \mathcal{T} \setminus \{I\}$. □

Proof. This follows directly from Definition 3.3.7. Q.E.D.

Note that the overlapping criterion in this case corresponds to the concept of overlapping given by Definition 3.3.8.

In the case in which $\mathcal{T}$ is a finite chain — to fix ideas, let $\mathcal{T} = \{0, 1, \ldots, R-1\}$, where $R \geq 2$ is a finite integer— it is clear that $Y_\tau(f) = X_{\tau+1}$, for $\tau \in \mathcal{T} \setminus \{R-1\}$. Hence, given $\tau \in \mathcal{T} \setminus \{0\}$, the overlap criterion in (4.76) can be written as

$$\perp_\tau(\{f_\alpha\}) = \begin{cases} I_\perp, & \text{if } \bigcap_\alpha \{X_{\tau'}(f_\alpha) \mid X_{\tau'}(f_\alpha) \neq \emptyset\} \neq \emptyset, \text{ for all } 1 \leq \tau' \leq \tau \\ O_\perp, & \text{otherwise} \end{cases}, \tag{4.79}$$

while the flat τ-hyperconnectivity class in (4.77) can be written as:

$$\mathcal{H}_\tau = \{f \in \text{Fun}(E, \mathcal{T}) \mid X_{\tau'}(f) \in \mathcal{C}_{\tau'-1}, \text{ for all } 1 \leq \tau' \leq \tau\}. \tag{4.80}$$

Obviously, the flat τ-hyperconnectivity class associated with a connectivity class $\mathcal{C}$ is given simply by $\mathcal{H}_\tau = \{f \in \text{Fun}(E, \mathcal{T}) \mid X_{\tau'}(f) \in \mathcal{C}, \text{ for all } 1 \leq \tau' \leq \tau\}$ (it follows that, in this case, a function is flat hyperconnected if and only if it has a single connected regional maximum, see Section 4.2.4).

These observations form the basis for the following result, which shows that fuzzy topographic connectivity is also an example of flat hyperconnectivity.

4.4.5 Proposition. Let E be a finite subset of $\mathbb{Z}^n$, and let $\mathcal{T} = \{0, 1, \ldots, R-1\}$, where $R \geq 2$ is a finite integer. A function $f \in \text{Fun}(E, \mathcal{T})$ is topographically connected if and only if it is flat hyperconnected in $\text{Fun}(E, \mathcal{T})$, with respect to the family $\{\mathcal{C}_\tau \mid \tau \in \mathcal{T} \setminus \{R-1\}\}$ of connectivity classes in $\mathcal{P}(E)$, given by

$$\mathcal{C}_\tau = \{A \in \mathcal{P}(E) \mid A \text{ is connected in the graph } G = (E, X_{\tau+1}(\sigma_f))\}, \tag{4.81}$$

for $\tau \in \mathcal{T} \setminus \{R-1\}$, where σ_f is the $\mathcal{T}$-fuzzy relation given by (3.16). □

PROOF. This follows directly from the preceding observations and Proposition 3.4.11. Q.E.D.

Note that the overlapping criterion here corresponds to the concept of overlapping given by Definition 3.4.12.

4.4.2 Hyperconnectivity Openings

Hyperconnectivity openings are defined in a similar fashion to connectivity openings. Given a hyperconnectivity class $\mathcal{H}$ in a lattice $\mathcal{L}$, we define subclasses $\mathcal{H}_x \subseteq \mathcal{H}$ by

$$\mathcal{H}_x = \{H \in \mathcal{H} \mid H = O \text{ or } H \geq x\}, \quad x \in \mathcal{S}. \tag{4.82}$$

The hyperconnectivity openings associated with $\mathcal{H}$ are given by

$$\eta_x(A) = \bigvee\{H \in \mathcal{H}_x \mid H \leq A\} = \bigvee\{H \in \mathcal{H} \mid x \leq H \leq A\}, \quad A \in \mathcal{L}, \tag{4.83}$$

for every $x \in \mathcal{S}$. By comparing (4.83) to (4.8), it is clear that, if $\mathcal{H}$ is a connectivity class, then hyperconnectivity openings reduce to standard connectivity openings.

As in the standard connectivity case, we can also define a hyperreconstruction operator. Given a marker $M \in \mathcal{L}$, the hyperreconstruction $\vartheta(A \mid M)$ of $A \in \mathcal{L}$ given M is defined by:

$$\vartheta(A \mid M) = \bigvee_{x \leq M} \eta_x(A). \tag{4.84}$$

Being a supremum of openings, the operator $\vartheta(\cdot \mid M)$ is an opening on $\mathcal{L}$, for a fixed marker $M \in \mathcal{L}$.

Due to the weakening introduced by the notion of hyperconnectivity, hyperconnectivity openings lose some of the nice properties that connectivity openings satisfy. This is discussed next.

Recall that $\langle \mathcal{M} \mid \vee \rangle$ denotes the family that is sup-generated by $\mathcal{M}$; i.e., the family consisting of all elements of $\mathcal{L}$ obtained by taking suprema of elements of $\mathcal{M}$. It is easy to verify that the invariance domain of η_x is given by

$$\mathrm{Inv}(\eta_x) \;=\; \langle \mathcal{H}_x \mid \vee \rangle. \tag{4.85}$$

If $\mathcal{H}$ is a connectivity class, then $\mathrm{Inv}(\eta_x) = \mathcal{H}_x$; i.e., A is invariant to η_x if and only if $A = O$ or A is connected and $x \leq A$. For a connectivity class, this implies that the subclasses $\mathcal{H}_x$ are sup-closed, for all $x \in \mathcal{S}$. This property is not in general true for hyperconnectivity classes. In addition, there is no guarantee that, in general, the hyperconnectivity opening $\eta_x(A)$ will be hyperconnected; i.e., it may happen that $\eta_x(A) \notin \mathcal{H}$, for some $x \in \mathcal{S}$. As a matter of fact, each of these two properties *characterize* connectivity classes. This is shown by the following proposition.

4.4.6 Proposition. Let $\mathcal{H}$ be a hyperconnectivity class in a lattice $\mathcal{L}$. The following assertions are equivalent:

(a) $\eta_x(\mathcal{L}) \subseteq \mathcal{H}$, for every $x \in \mathcal{S}$,

(b) $\mathcal{H}_x$ is sup-closed, for every $x \in \mathcal{S}$,

(c) $\mathcal{H}$ is a connectivity class in $\mathcal{L}$. □

PROOF. $(a) \Rightarrow (b)$: For a given $x \in \mathcal{S}$, let $\mathcal{H}' \subseteq \mathcal{H}_x$. We need to show that $\bigvee \mathcal{H}' \in \mathcal{H}_x$. Note that $\mathcal{H}' \subseteq \text{Inv}(\eta_x)$, which implies that $\bigvee \mathcal{H}' \in \text{Inv}(\eta_x)$, since $\text{Inv}(\eta_x)$ is sup-closed. Now, if $\mathcal{H}'$ is empty or $\mathcal{H}' = \{O\}$, then $\bigvee \mathcal{H}' = O$ and we are done. Otherwise, we have that $x \leq \bigvee \mathcal{H}'$, so $x = \eta_x(x) \leq \eta_x(\bigvee \mathcal{H}') = \bigvee \mathcal{H}' \in \mathcal{H}$, from the assumption that $\eta_x(\mathcal{L}) \in \mathcal{H}$. It follows that $\bigvee \mathcal{H}' \in \mathcal{H}_x$, as required.

$(b) \Rightarrow (c)$: Axioms (i) and (ii) of a connectivity class are clearly satisfied. In order to show axiom (iii), consider a family $\{H_\alpha\}$ in $\mathcal{H}$, such that $\bigwedge H_\alpha \neq O$. Since $\mathcal{S}$ is sup-generating, we can pick $x \leq \bigwedge H_\alpha$, for some $x \in \mathcal{S}$. This implies that $x \leq H_\alpha \Rightarrow H_\alpha \in \mathcal{H}_x$, for all α. Since $\mathcal{H}_x$ is sup-closed, it follows that $\bigvee H_\alpha \in \mathcal{H}_x \Rightarrow \bigvee H_\alpha \in \mathcal{H}$.

$(c) \Rightarrow (a)$: This is obvious. Q.E.D.

This proposition shows that a hyperconnectivity class that is not a connectivity class will fail to satisfy both (a) and (b). Nevertheless, hyperconnectivity openings satisfy properties that resemble those of connectivity openings. In particular, we have the following result.

4.4.7 Proposition. Let $\mathcal{L}$ be a lattice with sup-generating family $\mathcal{S}$, furnished with an overlap criterion $\perp$, and let $\mathcal{H}$ be a hyperconnectivity class in $\mathcal{L}$. If $\eta_x(A), \eta_y(A) \in \mathcal{H}$, for some $x, y \in \mathcal{S}(A)$, then $\eta_x(A) \perp \eta_y(A) \Rightarrow \eta_x(A) = \eta_y(A)$. □

PROOF. The proof is similar to the one given in Theorem 4.1.9. From axiom (iii) of hyperconnectivity classes, we have that $\eta_x(A), \eta_y(A) \in \mathcal{H}$ and $\eta_x(A) \perp \eta_y(A)$ imply $H = \eta_x(A) \bigvee \eta_y(A) \in \mathcal{H}$. Moreover, since $\eta_x(A), \eta_y(A) \leq A$, we have that $H \leq A$. Now, $x \leq A \Rightarrow x \leq \eta_x(A) \Rightarrow x \leq H \Rightarrow H \leq \eta_x(A) \Rightarrow \eta_y(A) \leq \eta_x(A)$. The reverse inclusion is shown analogously. Q.E.D.

The previous proposition says that, if $\eta_x(A)$ and $\eta_y(A)$ are hyperconnected, then they are either equal or do not overlap, according to the overlapping criterion $\perp$.

In similar fashion to the definition of connected components (see Definition 4.1.5), we can define the concept of hyperconnected components.

4.4.8 Definition. Let $\mathcal{L}$ be a lattice, furnished with a hyperconnectivity class $\mathcal{H}$, and let $A \in \mathcal{L}$. A *hyperconnected component* of A is a nonzero element $H \in \mathcal{H}$ such that: (a) $H \leq A$, and (b) there is no $H' \in \mathcal{H}$ such that $H \leq H' \leq A$. △

Hyperconnected components partition the object to which they belong, in the following sense. Given an element $A \in \mathcal{L}$, let $\mathcal{H}(A)$ be the family of all hyperconnected components of A. The following properties can be easily verified:

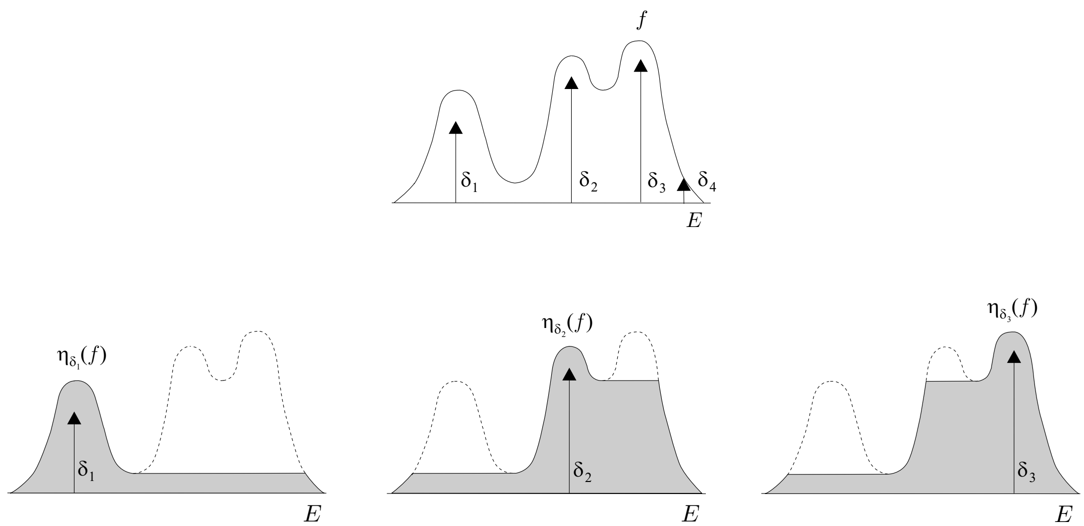


Figure 4.24: An example of hyperconnectivity openings, assuming flat hyperconnectivity on $\mathrm{Fun}(E,\overline{\mathbb{R}})$, associated with the Euclidean topological connectivity on $E = \mathbb{R}$. Top: Original function f and four sup-generators. Bottom: Hyperconnected components of f, which correspond to functions $\eta_{\delta_1}(f)$, $\eta_{\delta_2}(f)$ and $\eta_{\delta_3}(f)$. The function $\eta_{\delta_4}(f)$, not shown, corresponds to the original function f; hence, it is not hyperconnected.

(*i*) $H_1 = H_2$ or $H_1 \not\perp H_2$, for every $H_1, H_2 \in \mathcal{H}(A)$,

(*ii*) $A = \bigvee_{H_\alpha \in \mathcal{H}(A)} H_\alpha$.

Hence, the "partition" of hyperconnected components behaves in a similar fashion to ordinary partitions, since distinct zones (i.e., hyperconnected components) do not overlap, and their supremum reconstitutes the original object.

Now, let $\mathcal{Q}(A) = \{\eta_x(A) \mid x \leq A\} \cap \mathcal{H}$. If $\mathcal{H}$ is a connectivity class, then it is clear that $\mathcal{Q}(A) = \mathcal{H}(A)$ (see Proposition 4.1.6 and the discussion that follows). For arbitrary hyperconnectivity classes, it is easy to see that $\mathcal{Q}(A) \subseteq \mathcal{H}(A)$ (the argument is similar to the one given in the proof of Proposition 4.1.6). In other words, for $x \leq A$, if $\eta_x(A)$ is hyperconnected, then $\eta_x(A)$ is a hyperconnected component of A.

Fig. 4.24 illustrates these concepts. In this example, we consider the case of flat hyperconnectivity. The original function f has three hyperconnected components, each one associated with a distinct regional maximum of f. Note that $\eta_{\delta_4}(f) = f$, so that $\eta_{\delta_4}(f)$ is not hyperconnected.

For the example depicted in Fig. 4.24, we have that $\mathcal{Q}(f) = \mathcal{H}(f)$. However, it is possible in some cases to have the strict inequality $\mathcal{Q}(A) \subset \mathcal{H}(A)$, for some element $A \in \mathcal{L}$; i.e., there might exist a hyperconnected component H of A such that $H \neq \eta_x(A)$, for all $x \in \mathcal{S}$. For instance, consider $\mathcal{L} = \mathcal{P}(E)$, where $E = \{a, b, c, d\}$, with the points as sup-generators, and let $\mathcal{P}(E)$ be furnished with the overlap criterion

$$\perp(\{A_\alpha\}) = \begin{cases} I_\perp, & \text{if } |\bigcap A_\alpha| \geq 2 \\ O_\perp, & \text{otherwise} \end{cases}. \tag{4.86}$$

The family

$$\mathcal{H} = \{\emptyset, \{a\}, \{b\}, \{c\}, \{d\}, \{a, b\}, \{b, c\}, \{c, d\}\}, \tag{4.87}$$

defines a hyperconnectivity class in $\mathcal{P}(E)$. Clearly

$$\mathcal{Q}(E) = \{\{a, b\}, \{c, d\}\}, \tag{4.88}$$

whereas

$$\mathcal{H}(E) = \{\{a, b\}, \{b, c\}, \{c, d\}\}. \tag{4.89}$$

The hyperconnected component $\{b, c\}$ does not arise from any of the hyperconnectivity openings; note that $\eta_{\{b\}}(E) = \{a, b, c\} \notin \mathcal{H}$ and, similarly, $\eta_{\{c\}}(E) = \{b, c, d\} \notin \mathcal{H}$.

The previous counterexample gives another instance when $\eta_x(A)$ fails to be hyperconnected for some sup-generator $x \leq A$. It is easy to see that this happens if and only if there are distinct hyperconnected components of A that are marked simultaneously by x. Such a sup-generator can be considered to belong to an "uncertainty" region, which is in between hyperconnected components. This idea is made precise in the next subsection.

4.4.3 Z-operators and Segmentation by Similarity Zones

We now introduce the notion of Z-operators, associated with a hyperconnectivity class, and show that these operators can be effectively used for segmentation.

Consider a lattice $\mathcal{L}$ with sup-generating family $\mathcal{S}$, furnished with a hyperconnectivity class $\mathcal{H}$, and let $A \in \mathcal{L}$. We define the following relation between sup-generators:

$$x \overset{A}{\sim} y \quad \text{if} \quad \eta_x(A) = \eta_y(A), \quad x, y \in \mathcal{S}, \tag{4.90}$$

where $\{\eta_x \mid x \in \mathcal{S}\}$ are the hyperconnectivity openings associated with $\mathcal{H}$. It is obvious that $\overset{A}{\sim}$ is an equivalence relation on $\mathcal{S}$. In particular, this relation partitions the sup-generators into equivalence classes.

Based on the equivalence relation $\overset{A}{\sim}$, we can define the following class of operators on $\mathcal{L}$:

$$\zeta_x(A) = \begin{cases} \bigvee\{y \in \mathcal{S} \mid y \overset{A}{\sim} x\}, & x \leq A \\ O, & \text{otherwise} \end{cases}, \quad x \in \mathcal{S}. \tag{4.91}$$

These are called the *Z-operators* associated with the hyperconnectivity class $\mathcal{H}$. Note that Z-operators are anti-extensive, but are not in general increasing.

We show next that Z-operators provide a method for partitioning binary images. Consider the lattice $\mathcal{L} = \mathcal{P}(E)$ with the points as sup-generators, and let $\mathcal{H}$ be a hyperconnectivity class in $\mathcal{P}(E)$. It is easy to verify that, given a set $A \in \mathcal{P}(E)$, the mapping $z_A\colon A \to \mathcal{P}(E)$, given by

$$z_A(v) = \zeta_{\{v\}}(A) = \bigcup\{\{w\} \subseteq E \mid \{w\} \overset{A}{\sim} \{v\}\}, \quad v \in A, \tag{4.92}$$

defines a set partition of A. We refer to z_A as the *segmentation by similarity zones* of A. Note that each similarity zone $Z = z_A(v)$, for $v \in A$, of the partition z_A is associated with a characteristic set $F \subseteq E$, such that $\eta_{\{w\}}(A) = F$, for all $w \in Z$. Distinct zones are associated with distinct characteristic sets. If the characteristic set F is hyperconnected, then F is a hyperconnected component of A. In this case, the similarity zone is said to be a *component zone* of A. A zone associated with a non-hyperconnected set F is called a *transition zone* of A. These zones correspond to transition zones that are "in-between" hyperconnected components. These ideas are illustrated in Fig. 4.25 (see also Fig. 4.9, which depicts the two 2-connected components of A).

The following result shows that Z-operators reduce to standard connectivity openings, when $\mathcal{H}$ is a connectivity class.

4.4.9 Proposition. Let $\mathcal{L}$ be a lattice with sup-generating family $\mathcal{S}$, furnished with a hyperconnectivity class $\mathcal{H}$. If $\mathcal{H}$ is a connectivity class in $\mathcal{L}$, then $\zeta_x = \gamma_x$, for $x \in \mathcal{S}$, where $\{\gamma_x \mid x \in \mathcal{S}\}$ are the connectivity openings associated with $\mathcal{H}$. □

Proof. Let $A \in \mathcal{L}$. If $x \not\leq A$, the result is trivial. So, let $x \leq A$. For $y \overset{A}{\sim} x$, we have that $y \leq \gamma_y(A) = \gamma_x(A)$. It follows that $\zeta_x(A) = \bigvee\{y \in \mathcal{S} \mid y \overset{A}{\sim} x\} \leq \gamma_x(A)$. Conversely, for $y \leq \gamma_x(A)$, we have that $y \leq A \Rightarrow y \leq \gamma_y(A) \Rightarrow \gamma_x(A) \bigwedge \gamma_y(A) \geq y \neq O \Rightarrow \gamma_x(A) = \gamma_y(A) \Rightarrow y \overset{A}{\sim} x \Rightarrow y \leq \zeta_x(A)$, so $\gamma_x(A) = \bigvee\{y \in \mathcal{S} \mid y \leq \gamma_x(A)\} \leq \zeta_x(A)$. Hence, $\zeta_x(A) = \gamma_x(A)$. Q.E.D.

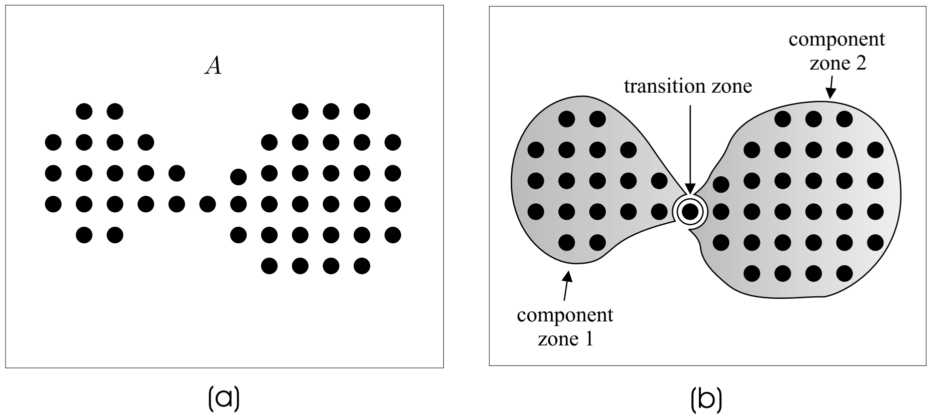


Figure 4.25: An example of a segmentation by similarity zones. The assumed hyperconnectivity class is the graph-theoretic k-connectivity class of Example 4.4.3(b), with $k = 2$. The usual 8-adjacency connectivity is assumed as the base connectivity class. (a) A binary image $A \in \mathcal{P}(E)$. (b) Segmentation by similarity zones of A.

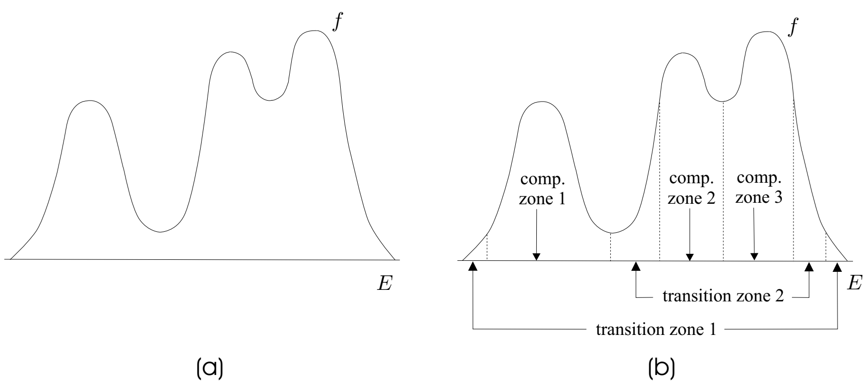


Figure 4.26: An example of segmentation by similarity zones of a function f. The assumed hyperconnectivity class is the flat hyperconnectivity class in $\mathrm{Fun}(E, \overline{\mathbb{R}})$, associated with the Euclidean topological connectivity on $E = \mathbb{R}$. (a) A grayscale function $f \in \mathrm{Fun}(E, \overline{\mathbb{R}})$. (b) Segmentation by similarity zones of f.

As a corollary, if $\mathcal{L} = \mathcal{P}(E)$ and $\mathcal{H}$ is a connectivity class in $\mathcal{L}$, the segmentation by similarity zones of an element $A \in \mathcal{P}(E)$ reduces to the PCC of A. Of course, this is a "hard" segmentation, in the sense that it has only component zones and no transition zones.

In the case of the general function lattice $\mathcal{L} = \mathrm{Fun}(E, \mathcal{T})$, Z-operators cannot be used directly to define a segmentation of a function $f \in \mathrm{Fun}(E, \mathcal{T})$. However, we obtain a segmentation of the support of f, if we introduce the following modification. Given a function $f \in \text{Fun}(E, \mathcal{T})$, consider the mapping $z_f \colon \Omega(f) \to \mathcal{P}(E)$, given by

$$z_f(v) = \bigcup \{\{w\} \subseteq \Omega(f) \mid \delta_{w,f(w)} \overset{f}{\sim} \delta_{v,f(v)}\}, \quad v \in \Omega(f). \tag{4.93}$$

It is not difficult to verify that z_f defines a set partition of the support $\Omega(f)$ of f (it is illustrative to compare (4.93) to (4.92)). We refer to z_f as the *segmentation by similarity zones* of f. Similarly as before, we say that a similarity zone $Z = z_f(v)$, $v \in \Omega(f)$, is a *component zone* of f, if $\eta_{\delta_{v,f(v)}}(f) \in \mathcal{H}$, for $v \in Z$, whereas Z is said to be an *transition zone* of f, if $\eta_{\delta_{v,f(v)}}(f) \notin \mathcal{H}$, for $v \in Z$.

An example of segmentation by similarity zones of a function f is depicted in Fig. 4.26, where flat hyperconnectivity is assumed. Fig. 4.27 shows an application of such a segmentation to the cornea cells image of Fig. 4.11, assuming flat hyperconnectivity as well. The image is first preprocessed by an open-close alternating sequential filter [77] in order to reduce noise. The solid regions depict component zones, whereas the unfilled regions depict transition zones. Most cells are represented accurately by component zones in the segmentation. The transition zones in the segmentation correspond to regions in between cells.

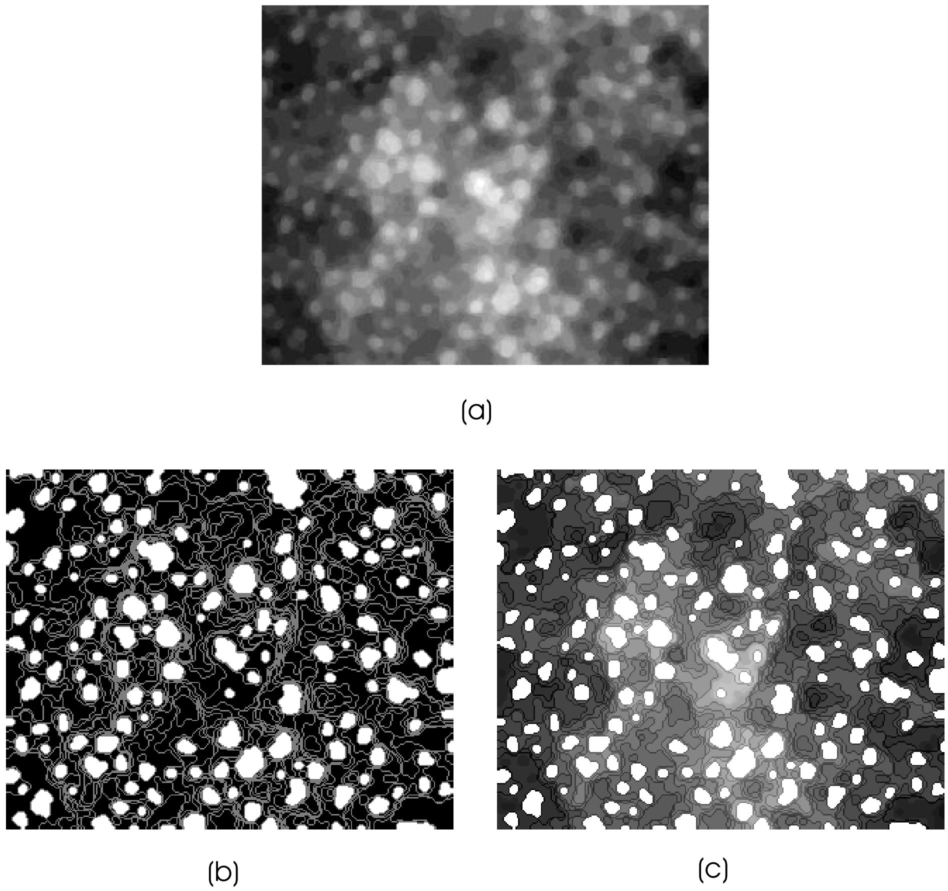


Figure 4.27: (a) The cornea cells image of Fig. 4.11 after preprocessing by an open-close alternating sequential filter. (b) Segmentation by similarity zones of the image in (a), assuming flat hyperconnectivity associated with the usual 8-adjacency connectivity. The solid regions depict component zones, whereas the unfilled regions depict transition zones. (c) The segmentation result in (b) superimposed on the original image in (a).

Chapter 5

Connected Operators

Connected operators have become very popular in recent years[19, 35, 37, 71, 73, 74, 87]. This is mainly due to the fact that these operators do not work at the pixel level, but rather at the level of the flat zones of an image, which are defined using connectivity criteria. A connected operator can remove boundaries, but cannot shift boundaries or introduce new ones. It therefore preserves contour/shape information, known to carry most of image content perceived by human observers.

In this chapter, we study connected operators in the framework of connectivity classes. In Section 5.1, we present the theory of binary connected operators, while in Section 5.2, we examine connected operators in the general function lattice case. Finally, in Section 5.3, we use a few examples taken from our previous work to demonstrate the effectiveness of connected operators in various image processing and analysis tasks, including mine detection in multispectral images, target detection and tracking in FLIR video sequences, and topology correction of 3-D brain MRI data.

5.1 Binary Connected Operators

In this section, we present basic facts about binary connected operators. For a more detailed exposition, the reader is referred to [35].

Consider the set lattice $\mathcal{L} = \mathcal{P}(E)$, with the points as sup-generators, and let $\mathcal{C}$ be an arbitrary connectivity class in $\mathcal{P}(E)$. For every set $A \in \mathcal{P}(E)$, let

$$z_A(x) = \begin{cases} \gamma_x(A), & \text{if} \quad x \in A \\ \gamma_x(A^c), & \text{if} \quad x \notin A \end{cases}, \tag{5.1}$$

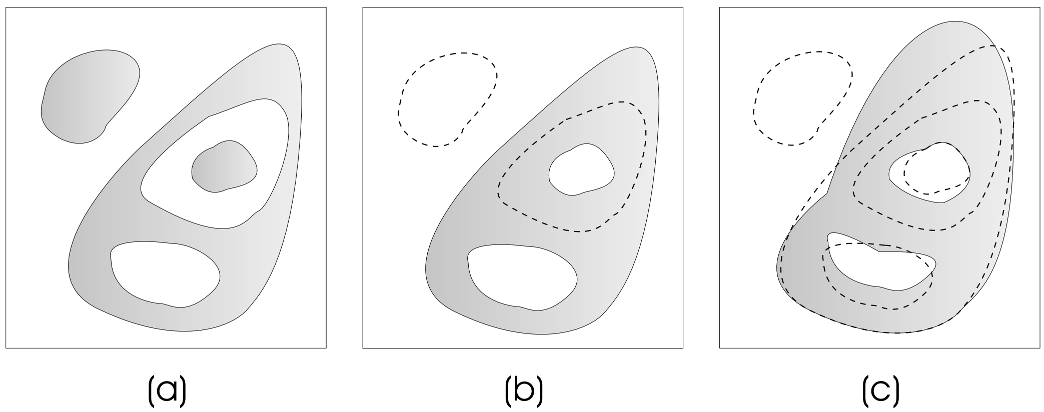


Figure 5.1: (a) A binary image A. (b) The output of a connected operator applied on A. (c) This image cannot possibly be the output of a connected operator applied on A.

where $\{\gamma_x \mid x \in E\}$ are the connectivity openings associated with $\mathcal{C}$. Clearly, z_A defines a partition of the image domain E, in the sense of Definition 4.1.7. This is called the *partition of flat zones* of A. The flat zones $\{z_A(x) \mid x \in E\}$ are either grains or pores of A. The following definition introduces the concept of a connected operator on $\mathcal{P}(E)$.

5.1.1 Definition. An operator ψ on lattice $\mathcal{P}(E)$ is said to be a *connected operator* if, for every $A \in \mathcal{P}(E)$, we have that $z_{\psi(A)}$ is coarser than z_A; i.e., for every $A \in \mathcal{P}(E)$, $z_A(x) \subseteq z_{\psi(A)}(x)$, for all $x \in E$. $\triangle$

Hence, an operator ψ on $\mathcal{P}(E)$ is connected if and only if, for each $A \in \mathcal{P}(E)$, the output $\psi(A)$ is constant over any flat zone Z of A; i.e., $Z \subseteq \psi(A)$ or $Z \subseteq (\psi(A))^c$. This implies that a binary connected operator can only remove entire grains, or fill entire pores, so that it acts by merging flat zones into larger ones. This means that a connected operator can only remove boundaries; it cannot shift, break, or introduce new boundaries (here, "boundary" is understood as the interface between flat zones). This is illustrated in Fig. 5.1.

The definition of a connected operator depends on the assumed connectivity class $\mathcal{C}$, which determines the partition of flat zones z_A in (5.1). Hence, ψ may be connected according to a given connectivity class $\mathcal{C}$, but not according to another connectivity class $\mathcal{C}'$. However, we have the following proposition (this result appears in [35, Prop. 7.4]; the proof given here is different from the one given in that reference).

5.1.2 Proposition. Let $\mathcal{C}$ and $\mathcal{C}'$ be two connectivity classes in $\mathcal{P}(E)$, such that $\mathcal{C} \subseteq \mathcal{C}'$. Every connected operator on $\mathcal{P}(E)$ according to $\mathcal{C}'$ is also connected according to $\mathcal{C}$. $\square$

PROOF. Let Z be a flat zone of $A \in \mathcal{P}(E)$, according to $\mathcal{C}$. Since $\mathcal{C} \subseteq \mathcal{C}'$, the flat zone Z must be contained in some flat zone Z' according to $\mathcal{C}'$. But ψ is connected according to $\mathcal{C}'$; i.e., $\psi(A)$ is constant over Z', and thus it is constant over Z, as well. Hence, ψ is connected according to $\mathcal{C}$. Q.E.D.

Below, we give examples of simple binary connected operators.

5.1.3 Example.

(a) The identity operator $A \mapsto A$, the complementation operator $A \mapsto A^c$, and the constant operators $A \mapsto \emptyset$ and $A \mapsto E$ are clearly connected operators on $\mathcal{P}(E)$, regardless of the assumed connectivity class.

(b) Binary connectivity openings $\{\gamma_x \mid x \in E\}$ are connected operators, according to the associated connectivity class $\mathcal{C}$ in $\mathcal{P}(E)$. This follows from the fact that $\gamma_x(A)$ keeps the foreground zone of A marked by x, if any, and merges all other foreground zones with the background zones.

(c) The binary reconstruction operator $\rho(\cdot \mid M)$ defines a connected operator, according to the associated connectivity class $\mathcal{C}$ in $\mathcal{P}(E)$. This follows from the fact that $\rho(A \mid M)$ keeps the foreground zones of A that intersect M, if any, and merges all other foreground zones with the background zones. ◊

The following result lists a number of ways for creating new connected operators from existing ones (for a proof, see [35, Prop. 7.5]).

5.1.4 Proposition. Let ψ, ϕ, and $\{\psi_i \mid i \in I\}$ be connected operators on $\mathcal{P}(E)$.

(i) The dual operator ψ^* is connected.

(ii) The composition $\psi\phi$ is connected.

(iii) The supremum $\bigvee \psi_i$ and infimum $\bigwedge \psi_i$ are connected. □

An interesting and useful class of binary connected operators are the so-called grain operators, introduced in [19, 35]. These connected operators act independently on each connected component of the foreground and the background, so that the output can be computed grain by grain and pore by pore.

Our discussion of grain operators is similar to that in [35], the major difference being that we focus on grain operators that act on the foreground and background separately, due to the practical importance of these operators. We start by defining *foreground criteria* and *background criteria* as mappings

$$u\colon\ \mathcal{P}(E) \to \{0,1\} \tag{5.2}$$

$$v\colon\ \mathcal{P}(E) \to \{0,1\}, \tag{5.3}$$

respectively. If $u(A) = 1$ (resp. $v(A^c) = 1$), then we say that A satisfies the foreground (resp. background) criterion imposed by u (resp. v). Using this concept, we define the *foreground trivial operator* and the *background trivial operator* on $\mathcal{P}(E)$ by

$$\iota_u(A) = \begin{cases} A, & \text{if } u(A) = 1 \\ \emptyset, & \text{otherwise} \end{cases} \tag{5.4}$$

$$\kappa_v(A) = \begin{cases} A, & \text{if } v(A^c) = 1 \\ E, & \text{otherwise} \end{cases}, \tag{5.5}$$

respectively, for $A \in \mathcal{P}(E)$. Note that these operators are dual to each other: $\iota_u^* = \kappa_u$. Note also that ι_u and κ_v are connected operators, regardless of the assumed connectivity class.

Now, let $\mathcal{C}$ be a connectivity class in $\mathcal{P}(E)$. The trivial operators allow us to define the *foreground grain operator* and the *background grain operator* by

$$\psi_u = \bigvee_{x \in E} \iota_u \gamma_x \tag{5.6}$$

$$\phi_v = \bigwedge_{x \in E} \kappa_v \varphi_x, \tag{5.7}$$

respectively, where $\{\gamma_x \mid x \in E\}$ are the connectivity openings associated with $\mathcal{C}$ and $\{\varphi_x \mid x \in E\}$ are the corresponding *connectivity closings*, given by $\varphi_x = \gamma_x^*$, for $x \in E$ (the connectivity closing $\varphi_x(A)$ extracts the pore of A marked by x). Note that

$$\psi_u^* = \left(\bigvee_{x \in E} \iota_u \gamma_x \right)^* = \bigwedge_{x \in E} (\iota_u \gamma_x)^* = \bigwedge_{x \in E} \iota_u^* \gamma_x^* = \bigwedge_{x \in E} \kappa_u \varphi_x = \phi_u; \tag{5.8}$$

i.e., the foreground grain operator and the background grain operator are dual to each other.

The action of foreground and background grain operators is made clear by the following alternative characterization (recall that $C \lessdot A$ means "C is a connected component of A").

5.1.5 Proposition. For $A \in \mathcal{P}(E)$,

$$\psi_u(A) = \bigcup\{C \lessdot A \mid u(C) = 1\} \tag{5.9}$$

$$\phi_v(A) = A \cup \bigcup\{C \lessdot A^c \mid v(C) = 0\}. \tag{5.10}$$

□

Proof. The expression for $\psi_u(A)$ is obvious from (5.4) and (5.6). For $\phi_v(A)$, we have that

$$\begin{aligned}\phi_v(A) = \psi_v^*(A) = (\psi_v(A^c))^c &= \Big(\bigcup\{C \mid C \lessdot A, v(C) = 1\}\Big)^c \\ &= A \cup \bigcup\{C \mid C \lessdot A^c, v(C) = 0\},\end{aligned} \tag{5.11}$$

as required. Q.E.D.

The previous result says that the foreground (resp. background) grain operator applies the foreground (resp. background) criterion on each grain (resp. pore) of a binary image $A \in \mathcal{P}(E)$ and keeps it or removes it (resp. fills it in) depending on whether it satisfies the corresponding criterion.

From (5.9) and (5.10), it is clear that ψ_u and ϕ_v are anti-extensive and extensive connected operators on $\mathcal{P}(E)$, respectively. Their most characteristic property, however, is that they are the only such operators that act independently on each grain and pore. The next result formalizes this statement.

5.1.6 Proposition.

(a) An anti-extensive connected operator ξ on $\mathcal{P}(E)$ is a foreground grain operator if and only if

$$\gamma_x \xi = \xi \gamma_x, \quad x \in E. \tag{5.12}$$

(b) An extensive connected operator ξ on $\mathcal{P}(E)$ is a background grain operator if and only if

$$\varphi_x \xi = \xi \varphi_x, \quad x \in E. \tag{5.13}$$

□

Proof. We show only part (a); part (b) follows by duality. The direct implication can be shown, in a straightforward manner, by using (5.9). To show the reverse implication, assume that ξ is an anti-extensive connected operator such that (5.12) holds. For $C \in \mathcal{C}$, define a foreground criterion u as follows (the value of u outside $\mathcal{C}$ is not relevant here):

$$u(C) = \begin{cases} 1, & \text{if } \xi(C) = C \\ 0, & \text{if } \xi(C) = \emptyset \end{cases}. \tag{5.14}$$

We show that $\xi = \psi_u$. First, note that $\xi\gamma_x = \psi_u\gamma_x$, since $\xi\gamma_x(A) = \gamma_x(A) = \psi_u\gamma_x(A)$, if $u(\gamma_x(A)) = 1$, and $\xi\gamma_x(A) = \emptyset = \psi_u\gamma_x(A)$, otherwise. In addition, note that $\gamma_x\psi_u = \psi_u\gamma_x$, as a direct consequence of the fact that ψ_u is a foreground grain operator. From (4.13) and (5.12), it follows that $\xi = \bigvee_{x\in E} \gamma_x\xi = \bigvee_{x\in E} \xi\gamma_x = \bigvee_{x\in E} \psi_u\gamma_x = \bigvee_{x\in E} \gamma_x\psi_u = \psi_u$. Q.E.D.

The previous result implies independent action on each grain and pore. For instance, in the case of a foreground grain operator ψ_u, Proposition 5.1.6(a) implies that, for any $A_1, A_2 \in \mathcal{P}(E)$ such that $\gamma_x(A_1) = \gamma_x(A_2) = C$, we have $\gamma_x\psi_u(A_1) = \gamma_x\psi_u(A_2) = \psi_u(C)$.

As a straightforward corollary of Proposition 5.1.6, the output of foreground and background grain operators can be computed grain by grain and pore by pore, respectively.

5.1.7 Corollary. For every $A \in \mathcal{P}(E)$,

$$\psi_u(A) = \bigcup_{x\in E} \psi_u\gamma_x(A) \tag{5.15}$$

$$\phi_v(A) = \bigcap_{x\in E} \phi_v\varphi_x(A). \tag{5.16}$$

□

Proof. From (4.13) and (5.12), we have that $\psi_u(A) = \bigcup_{x\in E} \gamma_x\psi_u(A) = \bigcup_{x\in E} \psi_u\gamma_x(A)$, which shows (5.15). Equation (5.16) follows by duality. Q.E.D.

Additional properties of foreground and background grain operators are given by the next proposition.

5.1.8 Proposition.

(a) The foreground grain operator ψ_u is idempotent. Moreover, it is increasing if u is increasing, in which case ψ_u is an opening.

(b) The background grain operator ϕ_u is idempotent. Moreover, it is increasing if v is increasing, in which case ϕ_v is a closing. □

PROOF. We show only part (a); part (b) follows by duality. Idempotence can be established as follows:

$$\begin{aligned}\psi_u\psi_u(A) &= \bigcup\{C \lessdot \psi_u(A) \mid u(C) = 1\} \\ &= \bigcup\{C \lessdot A \mid u(C) = 1\} \\ &= \psi_u(A). \end{aligned} \tag{5.17}$$

Now, assume that u is increasing; that is, $A \subseteq B \Rightarrow u(A) \leq u(B)$. Clearly, this implies that the foreground trivial operator ι_u is increasing. Since the connectivity opening γ_x is also increasing, and composition and union of increasing operators is increasing, we have that $\psi_u = \bigvee_{x\in E} \iota_u\gamma_x$ is increasing as well. Since ψ_u is also anti-extensive, it is an opening. Q.E.D.

Next, we give a few examples of foreground and background grain operators with increasing foreground and background criteria.

5.1.9 Example.

(a) For a given $x \in E$, the binary connectivity opening γ_x is a foreground grain operator, which is associated with the increasing foreground criterion given by $u(A) = 1$, if $x \in A$.

(b) The binary reconstruction operator $\rho(\cdot \mid M)$ is a foreground grain operator, which is associated with the increasing foreground criterion given by $u(A) = 1$, if $M \cap A \neq \emptyset$.

(c) Let s be the area feature on $\mathcal{P}(\mathbb{R}^2)$, defined in Section 6.5.3. For a given $k > 0$, consider the foreground criterion given by $u(A) = 1$, if $s(A) \geq k$. The associated foreground grain operator on $\mathcal{P}(\mathbb{R}^2)$ is known as *area opening*; the dual background grain operator is known as *area closing* (these operators are pseudo-openings and pseudo-closings, respectively, since the area feature is not always increasing; see Section 6.5.3). Similarly, one can define a *length opening* on $\mathcal{P}(\mathbb{R})$, a *volume opening* on $\mathcal{P}(\mathbb{R}^3)$, as well as the respective dual closings, and higher-dimensional analogs on $\mathcal{P}(\mathbb{R}^n)$, for $n \geq 4$. On the other hand, the *discrete* area feature on $\mathcal{P}(\mathbb{Z}^2)$, also defined in Section 6.5.3, is increasing, so that u is increasing, in this case. The associated foreground area operator on $\mathcal{P}(\mathbb{Z}^2)$ is known as *discrete area opening*; the dual background grain operator is known as *discrete area closing* (these operators are true openings and closings). Accordingly, one can define a *discrete length opening*

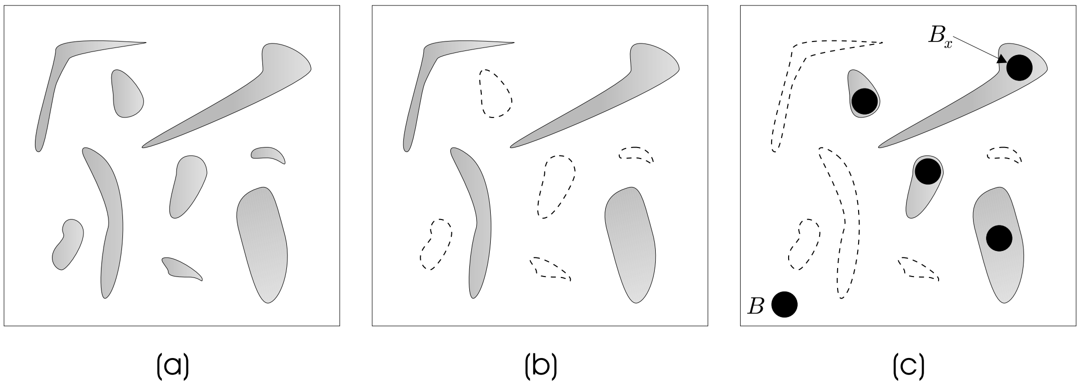


Figure 5.2: (a) Original image A. (b) Area opening $\check{\theta}_s(A)$. (c) Opening by reconstruction $\check{\theta}_B(A)$, with a disk structuring element B. The assumed connectivity is the usual Euclidean topological connectivity.

on $\mathcal{P}(\mathbb{Z})$, a *discrete volume opening* on $\mathcal{P}(\mathbb{Z}^3)$, as well as the respective dual closings, and higher-dimensional analogs on $\mathcal{P}(\mathbb{Z}^n)$, for $n \geq 4$.

(d) Let $E = \mathbb{R}^n$ or $\mathbb{Z}^n$. For a given $B \subseteq E$, consider the foreground criterion $u(A) = 1$, if B fits in A; i.e., $B_x \subseteq A$, for some $x \in E$. This criterion is increasing. The associated foreground grain operator on $\mathcal{P}(E)$ is known as *opening by reconstruction*; the dual background grain operator is known as *closing by reconstruction*. $\diamondsuit$

We denote the area opening and opening by reconstruction operators by $\check{\theta}_s$ and $\check{\theta}_B$, respectively. Similarly, $\check{\phi}_s$ and $\check{\phi}_B$ denote the area closing and the closing by reconstruction operators, respectively. These operators are very useful in practice [76, 89]. Fig. 5.2 depicts an illustration of area opening and opening by reconstruction. Note that the area opening tends to eliminate small components, while the opening by reconstruction tends to eliminate thin, elongated features.

Useful foreground and background grain operators can be designed with non-increasing criteria, in which case the operators are still (anti-)extensive and idempotent, in addition to being connected. As we saw, an example is the area opening on $\mathcal{P}(\mathbb{R}^2)$. Another example is associated with the perimeter criterion in $\mathbb{R}^2$, which is not increasing. The resulting foreground grain operator will remove all grains with perimeter less than some threshold. However, this is not an increasing operator, and therefore not an opening.

The term "opening by reconstruction" comes from the fact that, in practice, this foreground grain operator is usually implemented by means of reconstruction. In fact, we have the following result (see also [35, Prop. 12.4]).

5.1.10 Proposition. Let $E = \mathbb{R}^n$ or $\mathbb{Z}^n$, and let $\mathcal{C}$ be a translation-invariant connectivity class in $\mathcal{P}(E)$. If $B \in \mathcal{P}(E)$ is connected and contains the origin, then

$$\check{\theta}_B(A) = \rho(A \mid A \ominus B), \quad A \in \mathcal{P}(E). \tag{5.18}$$

□

Proof. First, note that the criterion "B fits in A" corresponds to the statement $A \ominus B \neq \emptyset$. Hence, by using Propositions 5.1.5 and 4.1.16, we need to show that

$$\bigcup\{C \lessdot A \mid C \ominus B \neq \emptyset\} = \bigcup\{C \lessdot A \mid C \cap (A \ominus B) \neq \emptyset\}. \tag{5.19}$$

Let $C \lessdot A$. We will show that $C \ominus B \neq \emptyset$ if and only if $C \cap (A \ominus B) \neq \emptyset$, which gives the desired result. The direct implication is obvious. To show the converse implication, first note that, since B contains the origin, we have that $C \ominus B = \{x \in E \mid B_x \subseteq C\} = \{x \in C \mid B_x \subseteq C\}$. On the other hand, we have that $C \cap (A \ominus B) = \{x \in C \mid B_x \subseteq A\}$. Now, let $x \in C \cap (A \ominus B)$; i.e., $x \in C$, with $B_x \subseteq A$. Note that $B_x \in \mathcal{C}$ and $x \in B_x \subseteq A$. Hence, $B_x \subseteq \gamma_x(A) = C$, which implies that $x \in C \ominus B$. Q.E.D.

Note that, when B is not connected, we still have the inequality $\check{\theta}_B(A) \subseteq \rho(A \mid A \ominus B)$.

In [35], H. Heijmans defines the following operator that simultaneously acts on the foreground and the background:

$$\zeta_{u,v}(A) = \bigcup\{C \mid C \lessdot A \text{ and } u(C) = 1 \text{ or } C \lessdot A^c \text{ and } v(C) = 0\}, \tag{5.20}$$

for $A \in \mathcal{P}(E)$, where, as before, u and v are foreground and background criteria, respectively. This is known as the *grain operator* and generalizes the foreground and background grain operators that we have considered above, since $\psi_u = \zeta_{u,1}$ and $\phi_v = \zeta_{1,v}$. Contrary to what might be expected, it is not true in general that $\zeta_{u,v} = \psi_u \phi_v$. Rather, by using Proposition 5.1.5, it is easy to see that

$$\zeta_{u,v}(A) \;=\; \psi_u(A) \cup [\phi_v(A) \setminus A] \;=\; \phi_v(A) \setminus [A \setminus \psi_u(A)], \tag{5.21}$$

for $A \in \mathcal{P}(E)$. However, if u and v are increasing, it can be shown that $\zeta_{u,v} = \psi_u \phi_v = \phi_v \psi_u$ (so that $\zeta_{u,v}$ is a strong filter, see [77]) if and only if $\zeta_{u,v}$ satisfies a property called stability [35].

The operator $\zeta_{u,v}$ is clearly a connected operator. Moreover, these operators form the only class of connected operators that act independently on each flat zone of a binary image.

The following result, which can be considered to be the analog of Proposition 5.1.6, makes this precise. To make the proof compact, we write $A(Z) = t$ to indicate that a set A is constant over a set Z, where $t = 1$, if $Z \subseteq A$, whereas $t = 0$, if $Z \subseteq A^c$.

5.1.11 Proposition. A connected operator ψ is a grain operator if and only if, for any $A_1, A_2 \in \mathcal{P}(E)$ such that C is a flat zone of both A_1 and A_2, then $\psi(A_1)(C) = \psi(A_2)(C)$. □

PROOF. "⇒": We have $\psi = \zeta_{u,v}$, so that

$$\psi(A_1)(C) = \psi(A_2)(C) = \begin{cases} u(C), & \text{if } C \text{ is a grain} \\ 1 - v(C), & \text{if } C \text{ is a pore} \end{cases}. \tag{5.22}$$

"⇐": Let $u(C) = \psi(C)(C)$ and $v(C) = 1 - \psi(C^c)(C)$, for $C \in \mathcal{C}$ (the value of u and v outside $\mathcal{C}$ does not matter here). It suffices to show that $\psi(A)(C) = \zeta_{u,v}(A)(C)$, for all $A \in \mathcal{P}(E)$, where C is any flat zone (grain or pore) of A. To fix ideas, assume that C is a grain of A. Then, $\zeta_{u,v}(A) = u(C) = \psi(C)(C)$. But C is a grain of both A and C. Thus, by hypothesis (with $A_1 = A$ and $A_2 = C$), we have that $\psi(A)(C) = \psi(C)(C)$. Therefore, $\psi(A)(C) = \zeta_{u,v}(A)(C)$, as required. The argument for the case when C is a pore of A is completely analogous. Q.E.D.

In other words, a connected operator ψ is a grain operator if and only if, for each flat zone C of A, we have that $\psi(A)(C)$ depends only on C. We remark that the above result corresponds to Proposition 8.4 in [35], though our statement and proof are more compact.

5.2 Function Connected Operators

In the previous section, we have used the term "flat zone" to signify either a grain or a pore in a binary image. It turns out that the concept of connected operator can be extended to the general function lattice case by generalizing the idea of a flat zone.

Given a function $f \in \text{Fun}(E, \mathcal{T})$, we can define a mapping $F : \mathcal{T} \to \mathcal{P}(E)$, given by

$$F(t) = \{x \in E \mid f(x) = t\}, \quad t \in \mathcal{T}. \tag{5.23}$$

For instance, in the binary case (e.g., when $\mathcal{T} = \{0, 1\}$), we have that $F(0)$ and $F(1)$ correspond to the image background and foreground, respectively. Let $\mathcal{C}$ be a connectivity class in $\mathcal{P}(E)$, and let

$$z_f(x) = \gamma_x(F(f(x))), \quad x \in E, \tag{5.24}$$

where $\{\gamma_x \mid x \in E\}$ are the connectivity openings associated with $\mathcal{C}$. Clearly, z_f defines a partition of the image domain E, in the sense of Definition 4.1.7, which we call the *partition of flat zones* of f. The flat zones $\{z_f(x) \mid x \in E\}$ are the largest connected regions of E where the value of f is constant; e.g., in the case of color images, a flat zone is a largest connected region that has a single color. It is easy to see that, in the binary case, (5.24) reduces to (5.1).

The definition of function connected operators can be essentially phrased in the exact same way as in the binary case.

5.2.1 Definition. An operator ψ on lattice $\mathrm{Fun}(E, \mathcal{T})$ is said to be a *connected operator* if, for every $f \in \mathrm{Fun}(E, \mathcal{T})$, we have that $z_{\psi(f)}$ is coarser than z_f; i.e., $z_f(x) \subseteq z_{\psi(f)}(x)$, for all $x \in E$. $\triangle$

In other words, an operator ψ on $\mathrm{Fun}(E, \mathcal{T})$ is connected if and only if, for each $f \in \mathrm{Fun}(E, \mathcal{T})$, the output $\psi(f)$ is constant over any flat zone of f. This implies that a function connected operator ψ acts by merging flat zones of an image f into larger ones. Clearly, a function connected operator cannot shift boundaries (i.e., move the interface between flat zones) or introduce new boundaries.

The following is the function analog of Proposition 5.1.2. The proof is omitted, being almost identical to the proof given for the binary result.

5.2.2 Proposition. Let $\mathcal{C}$ and $\mathcal{C}'$ be two connectivity classes in $\mathcal{P}(E)$, such that $\mathcal{C} \subseteq \mathcal{C}'$. Every connected operator on $\mathrm{Fun}(E, \mathcal{T})$ according to $\mathcal{C}'$ is also connected according to $\mathcal{C}$. $\square$

In the case in which $\mathcal{T}$ is a chain, function connected operators are known as *grayscale connected operators*. The most fundamental result about grayscale connected operators is that they can be obtained from binary connected operators by means of flat extension, discussed in Section 2.2. This is shown by the next result. For simplicity, we restrict ourselves to the case $\mathcal{T} = \overline{\mathbb{R}}$. The proof applies to arbitrary chains $\mathcal{T}$ as well (including discrete ones), with only minor modifications. To make the proof compact, we write $f(Z) = t$ to denote that f takes the constant value t over Z.

5.2.3 Proposition. If ψ is a binary connected operator on $\mathcal{P}(E)$, then the flat extension $\overline{\psi}$ is a grayscale connected operator on $\mathrm{Fun}(E, \overline{\mathbb{R}})$. $\square$

Proof. Let $f \in \mathrm{Fun}(E, \overline{\mathbb{R}})$, and Z be an arbitrary flat zone of f with value $t \in \overline{\mathbb{R}}$. To show the desired result, we need to show that $\overline{\psi}(f)$ is constant over Z (so that the partition of flat zones of $\overline{\psi}(f)$ is coarser than the partition of flat zones of f). First, note that, since $\overline{\mathbb{R}}$ is a chain,

$$f(Z) = t \;\;\Leftrightarrow\;\; Z \subseteq X_t(f) \text{ and } Z \subseteq [X_s(f)]^c, \text{ for } s > t, \tag{5.25}$$

where $X_t(f) = \{x \in E \mid f(x) \leq t\}$, for $t \in \overline{\mathbb{R}}$, are the threshold sets of f (it is easy to see that, in general, the reverse implication in (5.25) is not true for a lattice $\mathcal{T}$ that is not a chain). We also make use of the following fact: for a binary connected operator ψ and $A, B \in \mathcal{P}(E)$, with B being connected, we have that $B \subseteq A \Rightarrow B \subseteq \psi(A)$ and $B \subseteq A^c \Rightarrow B \subseteq [\psi(A)]^c$. From (5.25), it now follows that $f(Z) = t \Rightarrow Z \subseteq X_t(f) \Rightarrow Z \subseteq X_s(f)$, for $s < t$. Since Z is connected and ψ is a connected operator, this implies that $Z \subseteq \psi(X_s(f))$, for $s < t \Rightarrow Z \subseteq \bigcap_{s<t} \psi(X_s(f)) = X_t(\overline{\psi}(f))$, where we used (2.38). Similarly, we have that $f(Z) = t \Rightarrow Z \subseteq [X_r(f)]^c$, for $r > t \Rightarrow Z \subseteq [\psi(X_r(f))]^c$, for $r > t \Rightarrow Z \subseteq \bigcup_{s<r} [\psi(X_s(f))]^c = [\bigcap_{s<r} \psi(X_s(f))]^c = [X_r(\overline{\psi}(f))]^c$, for $r > t$, where we again used (2.38). From (5.25), this implies that $\overline{\psi}(f)(Z) = t$. Q.E.D.

This result first appeared in [75]; the proof above is an expanded version of the one given in that reference. This result guarantees that we can use tools related to binary connected operators, which we have encountered before, to build grayscale connected operators. Below, we give a few examples, where we use Examples 5.1.3 and 5.1.9 in conjunction with Proposition 5.2.3.

5.2.4 Example.

(a) The grayscale reconstruction operator $\overline{\rho}(\cdot \mid g)$, defined in (4.21), is a grayscale connected operator.

(b) The flat extensions of the binary area opening and the binary discrete area opening of Example 5.1.9(c) are grayscale connected operators, known as *grayscale area openings* [88]. Dually, grayscale connected operators known as *grayscale area closings* can be defined.

(c) Similarly, the flat extension of the binary opening by reconstruction of Example 5.1.9(d) is a grayscale connected operator, known as *grayscale opening by reconstruction*. Dually, a grayscale connected operator known as *grayscale closing by reconstruction* can be defined. ◊

The grayscale connected operators discussed above are very useful in image processing and analysis applications. This will be illustrated in the next section.

As a final remark, there is no definite formulation of a "grayscale grain operator," as has been also observed by H. Heijmans in [35]. The flat extension of binary grain operators can be characterized only loosely, as grayscale connected operators that act independently on image "maxima" and "minima" (so that if two different images coincide at a maximum, for instance, then the output for both should coincide there, too).

5.3 Applications

In this section, we present examples, based on our previous work, which illustrate the application of connected operators to image processing and analysis problems.

5.3.1 Landmine Detection in Multispectral Aerial Images

Automatic mine detection, a critical issue in battlefield management, is expected to provide accurate and reliable detection of mines embedded in clutter. In general, this is a very hard problem, which is facilitated by the acquisition of multispectral images. In [9], we described a procedure for automatic mine detection in multispectral data provided by the *Coastal Battlefield Reconnaissance and Analysis* (COBRA) program of the U.S. Navy. This method improved on previous efforts reported in [2]. As part of an ongoing research effort on automatic landmine detection, the work in [9] was later expanded in [3–5].

Fig. 5.3(a) depicts a band in one of the available six-band multispectral data sets. The targets have a characteristic grayscale profile and are set against a noisy background of sand and grass, which corresponds to the clutter. In [9], we developed a two-stage procedure to solve the problem of automatically extracting landmines from such data sets. The first stage of the proposed method consists of multispectral enhancement of the data by application of the so-called *Maximum Noise Fraction* (MNF) transform [29]. The second stage consists of a target detection algorithm, which is applied to each band of the enhanced data. The final detection result is obtained by majority voting among the results obtained for each band.

The MNF transform is used to reduce the effect of clutter and enhance the presence of targets. It is given by a $p \times p$ matrix (where p is the number of bands), which consists of the left eigenvectors of the matrix $\mathbf{C}_\eta \mathbf{C}_f^{-1}$, where $\mathbf{C}_f$ and $\mathbf{C}_\eta$ are the covariance matrices of the data and the clutter, respectively. Estimation of the data covariance matrix $\mathbf{C}_f$

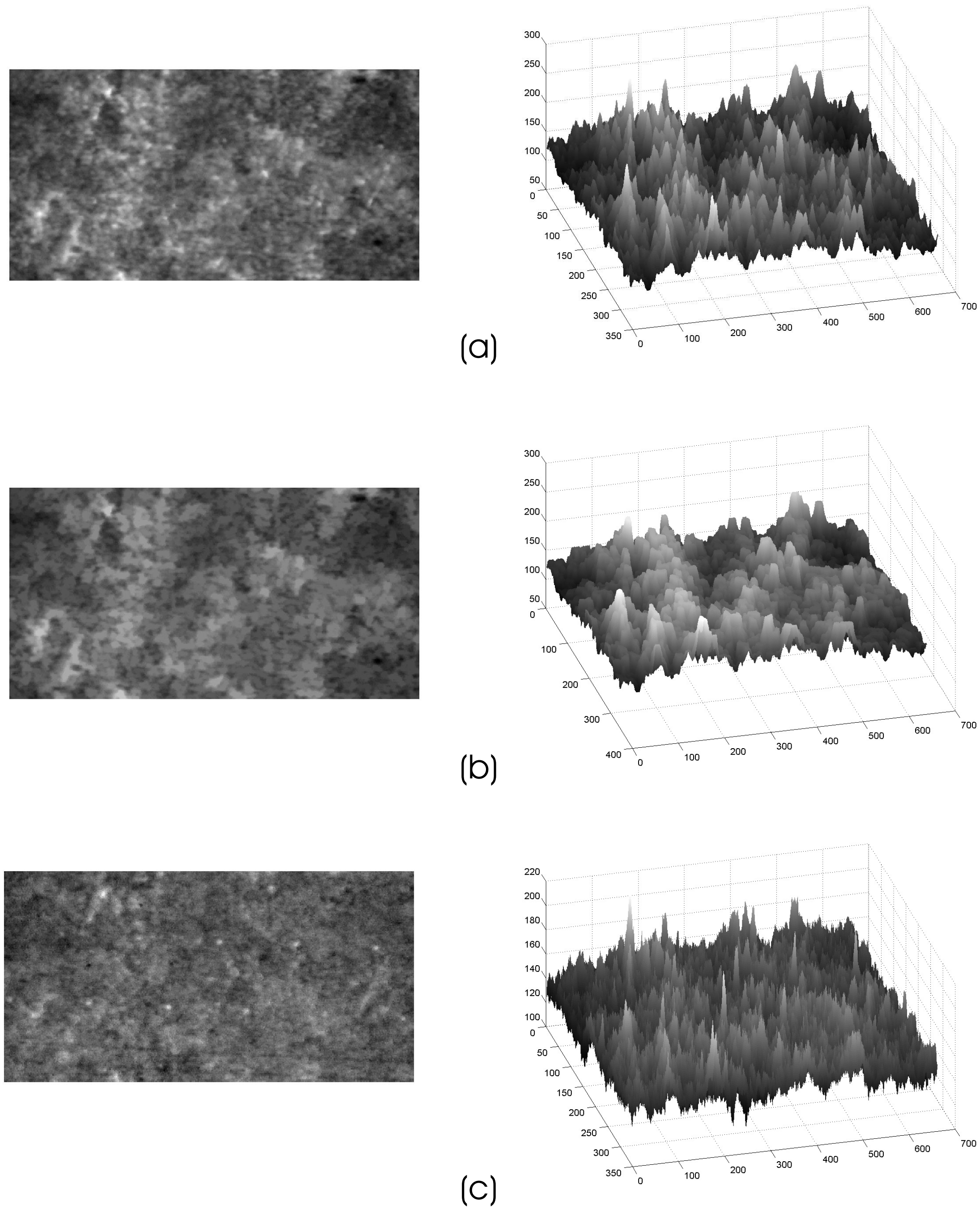


Figure 5.3: (a) Band 4 of original multispectral data. (b) Clutter approximation given by grayscale opening by reconstruction. (c) Band 4 of enhanced multispectral data after application of the MNF transform.

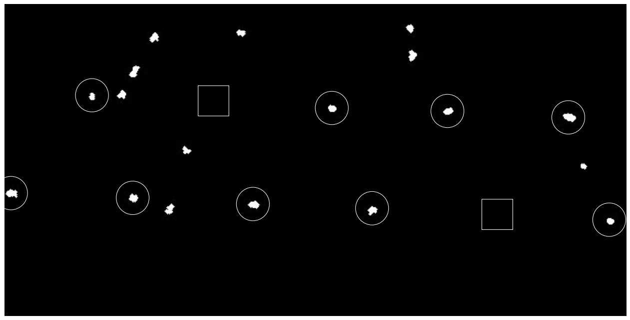


Figure 5.4: Final landmine detection result obtained with the proposed method. Circles indicate correctly detected targets, while squares indicate misdetections. Unmarked objects are false alarms.

can be easily done by means of standard covariance estimation techniques applied on the available data [85]. However, estimation of the clutter covariance matrix $\mathbf{C}_\eta$ necessitates an approximation of the clutter.

Therefore, estimation of the clutter is a critical step if the MNF transform is to be used. By exploring the fact that the landmines have a characteristic grayscale profile that approximates relatively small circular or elliptic spikes, we have found that grayscale opening by reconstruction provides a very good approximation of the clutter. The grayscale opening by reconstruction, by a conveniently sized disk structuring element, eliminates the landmines on each band, while minimizing distortion, thus obtaining a good approximation of the clutter. Figs. 5.3(b) and (c) depict the opening by reconstruction of the original band in (a) (i.e., the clutter approximation) and the corresponding band in the enhanced data, respectively. The connectivity used in this example was 8-adjacency connectivity.

Subsequent application of the second stage of the method, which consists of a target detection algorithm applied to the enhanced data, leads to excellent detection results (this stage also employs binary and grayscale openings by reconstruction). A low number of misdetections is observed, whereas only a small number of false alarms is introduced by the algorithm. This can be seen in Fig. 5.4, which displays the final result corresponding to the data set used in Fig. 5.3. The good results obtained demonstrate the effectiveness of binary and grayscale opening by reconstruction in this problem.

5.3.2 Target Detection and Tracking in FLIR Image Sequences

Automatic target detection and tracking in *forward-looking infrared* (FLIR) scenes is a difficult task, due to high variability of target types and background clutter and the way those can manifest themselves in images due to varying temperature and atmospheric conditions [43].

The particular FLIR image sequence data considered here was provided by the *U.S. Army Missile Command* (MICOM). The image sequences were obtained by means of a FLIR sensor mounted on an airborne platform. Fig. 5.5 shows three consecutive frames from one of the available image sequences. Note that the targets appear as bright features.

In [10], we have proposed a two-step method, based on connected operators, for automatic target detection and tracking in the FLIR image sequences provided by MICOM. Our method avoids the variability issue in FLIR scenes by not requiring any target modelling, in contrast, for instance, to pattern-theoretic approaches to the same problem [42]. The first step in our method consists of intraframe processing; i.e., processing of each frame separately. This step uses connected operators based on size and position criteria. The second step in our method involves interframe processing; i.e., processing across frames. Here, a connected operator based on a motion criterion is used.

The size and position criteria used in the intraframe processing step are spatial constraints on the targets of interest. The size criterion corresponds to the fact that the targets of interest have a maximal specified apparent size. In other words, very large features are likely to be background clutter, such as roads, buildings, etc. On the other hand, the position criterion requires that the targets of interest be situated away from the boundary of the *field of view* (FOV). This reflects the fact that targets of interest situated in the periphery of the sensor's FOV cannot be reliably detected (in addition, clutter is likely to extend beyond the FOV).

The size criterion is applied via a grayscale opening by reconstruction by an appropriately sized structuring element, which removes features of size smaller than some specified maximum size, followed by subtraction from the original frame. This corresponds to a background-removal type of operation. The position criterion is then applied, by means of grayscale reconstruction with a marker obtained in such a way that objects connected to the periphery of the image are eliminated (in addition, objects that are too small are not marked, and thus removed). The connectivity used throughout is digital 4-adjacency

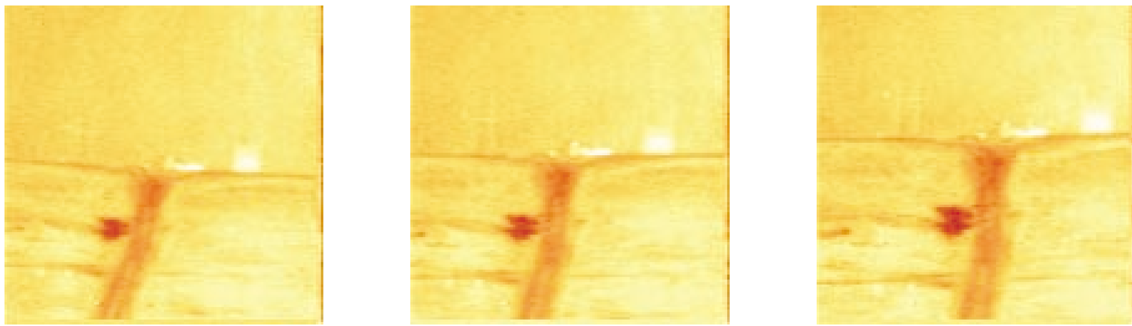


Figure 5.5: Three consecutive frames from one of the available FLIR sequences.

connectivity. Fig. 5.6 depicts the output of these operators on one of the bands. Note that the “heat” color map present in the original sequence is first removed, prior to application of the method.

After binarization of all frames, the interframe processing step of the method follows. This step imposes a motion criterion that reflects the fact that the targets of interest display a continuous trajectory across frames and have limited relative motion with respect to the FLIR sensor. The basic idea is to compute all spatio-temporal connected components (i.e., connected components in the 3-D volume formed by considering time as the third coordinate), and discard those that do not span a sufficient number of frames. However, the underlying connectivity needs to take into account temporal undersampling and sensor jitter, which makes the targets “jump” from frame to frame. The solution is to use a dilation-based connectivity class (see Section 4.3.1), where the base connectivity is 3-D digital 6-adjacency connectivity (connectivity across the faces of each volumetric image element), and the dilation is a translation invariant dilation by a 2-D structuring element of appropriate size. The larger the structuring element is, the more “jumpy” the trajectories are allowed to be (however, this also increases the false alarm rate). The action of the resulting motion connected operator is illustrated in Fig. 5.7.

The objects removed by the motion connected operator correspond to either residual clutter from the intraframe processing step, or targets that move too fast with respect to the sensor, which are not considered to be the primary targets of interest.

The overall method proves to be very effective and robust in detecting the targets of interest. An example is shown in Fig. 5.8, which presents the outputs of the intraframe processing step (intermediate result) and of the interframe processing step (final result)

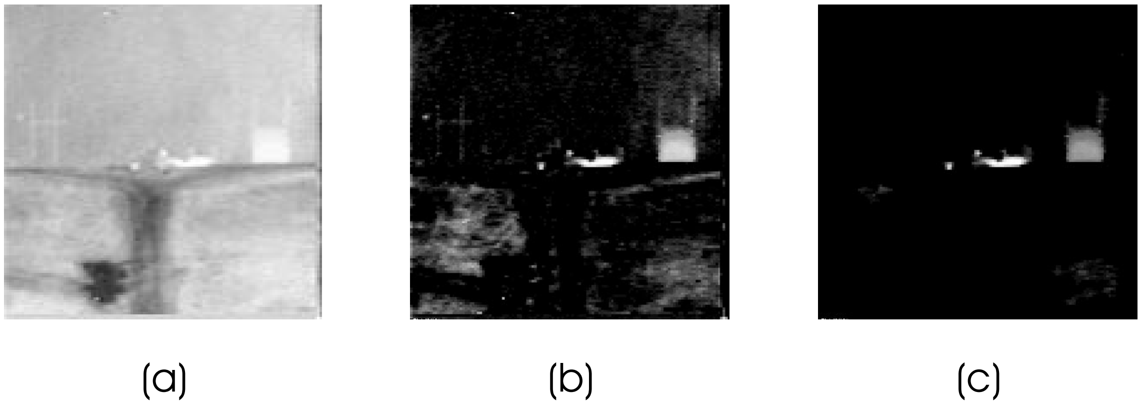


Figure 5.6: Intraframe processing. (a) Original frame. (b) Background removal using grayscale opening by reconstruction, according to size criterion. (c) Grayscale reconstruction of the image in (b) using a marker that reflects the position criterion.

that correspond to the consecutive frames depicted in Fig. 5.5. For display purposes, the contours of the detected targets are extracted by means of a morphological gradient operator [34] and superimposed on the original frames. Note that interframe processing is essential for removing leftover clutter from the intraframe processing step. Note also that the building to the right is not detected in the last frame, since it is located at the periphery of the FOV.

5.3.3 Topological Correction of Brain Cortical Surfaces

One of the difficulties in automatic segmentation and mapping of the brain cortex is establishing of the correct topology of the cortical surface [54, 81, 82, 94]. Ideally, the combined hemispheres of the brain, joined by the bridge across the *corpus callosum*, should map homeomorphically to a sphere. The problem encountered by several research groups is that it is hard to simultaneously create an accurate representation of the cortex surface (or the white matter/gray matter interface) and guarantee the correct topology. The most important difficulty is the creation of handles (like that of a coffee mug), with associated tunnels, in the representation of the surface. These features are not topologically consistent with a sphere and make it impossible to establish a homeomorphism.

We have studied a new approach, based on connected operators, to the problem of generating topologically correct surfaces. This method fits in the framework of the cortical reconstruction algorithm of [94] and develops a new algorithm in the spirit of that of [81, 82]. Our work has led to a related method that uses some of the ideas described here [30, 31].

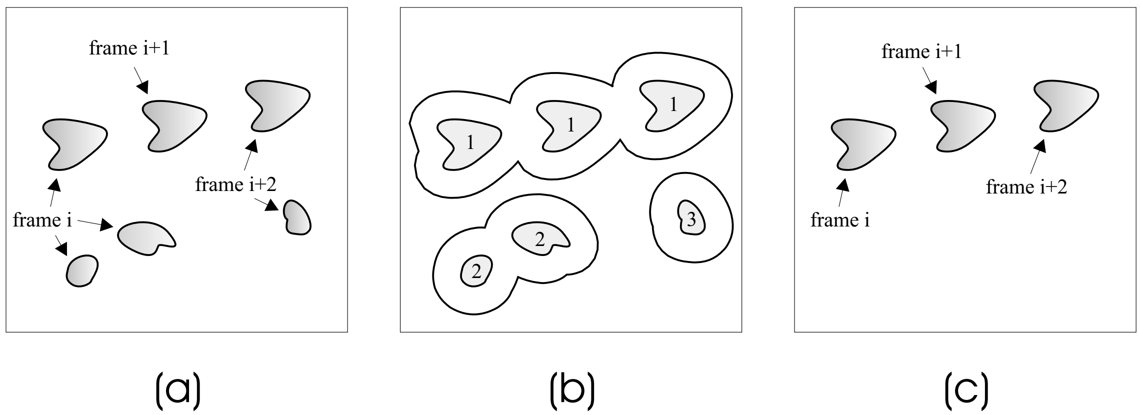


Figure 5.7: Interframe processing. (a) Objects in three consecutive frames. (b) Spatio-temporal connected components, according to the dilation-based connectivity class. (c) Output of the motion connected operator, which eliminates spatio-temporal connected components that do not span enough frames.

Our approach consists of an alternating sequential composition of thinnings and thickenings, which we refer to as the *ASTT filter*. This filter removes handles and fills in tunnels in the membership function of a 3-D fuzzy segmentation of the white matter of the brain. Each thinning is a supremum of 2-D grayscale area openings of a given size parameter, which are applied on slices of the volume along a given axis. The supremum is taken over a uniform sampling of all possible axis directions. The thinnings are responsible for removing handles whose section area is less than the given size. Dually, each thickening is an infimum of 2-D grayscale area closings of a given size along a given axis. The infimum is taken over the same uniform sampling of directions mentioned above. The thickenings fill in tunnels whose section area is less than the given size.

The proposed thinnings and thickenings are applied in serial, alternating fashion, where one starts with a small size and then increases it, while switching between thinnings and thickenings, until the desired result is achieved. The motivation behind this approach is that it is possible for a given topological hole to be associated with a large tunnel and a comparatively thin handle. In this case, it is advantageous to break the handle instead of filling in the tunnel, since the former implies a smaller topological correction. The converse applies to the case of a tight tunnel surrounded by a bulky handle. Alternating composition of thinnings and thickenings of increasing size guarantees that the topological corrections made lead to a small amount of distortion.

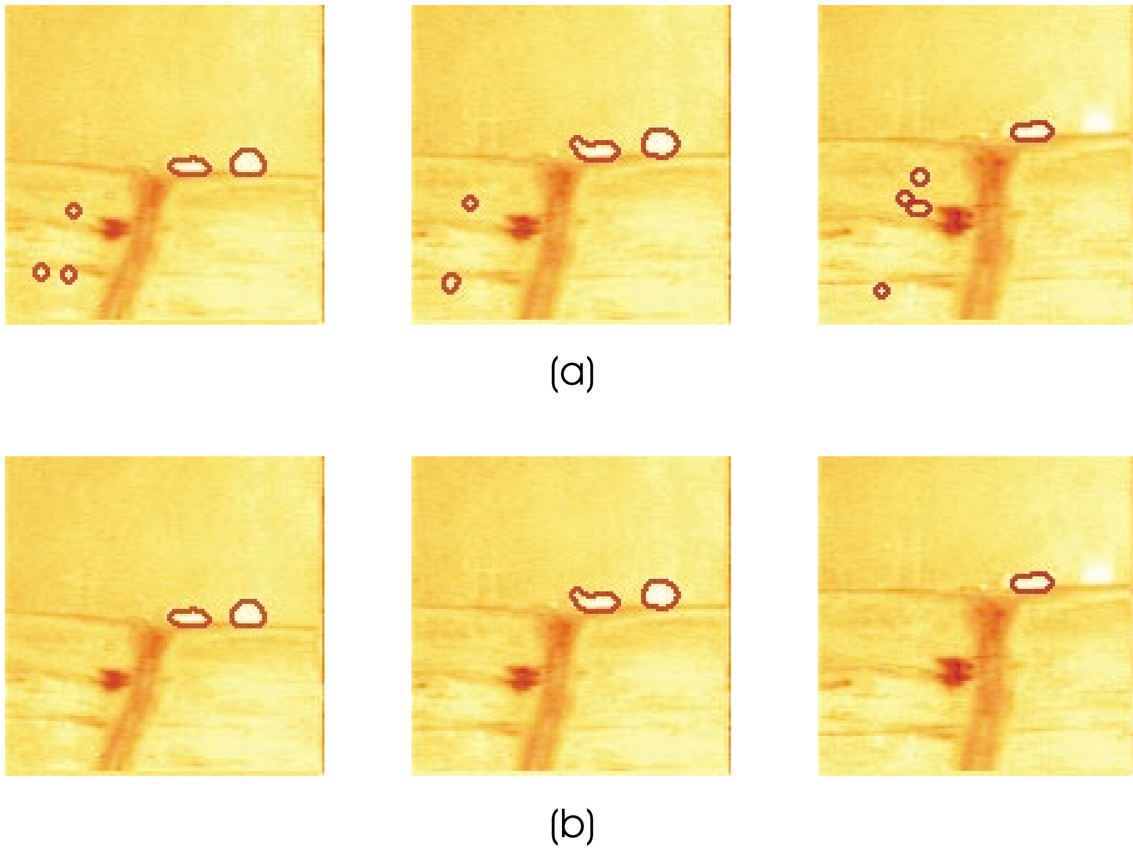


Figure 5.8: Consecutive frames from (a) the intraframe processing step, and (b) the interframe processing step.

This approach is innovative in several respects. First, it operates on both the background and the foreground of the membership function; in this way, handles and tunnels are treated equivalently. Second, it operates on the grayscale values of the membership function; this preserves subvoxel resolution, unlike binary methods that have been reported in the literature. Third, as discussed above, it operates in an alternating sequential fashion, minimizing overall distortion. Finally, a uniform sampling of orientations is considered for each thinning/thickening so that the method is insensitive to orientation selection, unlike other methods found in the literature.

To show the effectiveness of the proposed method, we have compared our results to the results obtained using only median filters, which is the approach used in [94]. This approach consists of iterating a $3 \times 3 \times 3$ box median filter until a topologically correct surface is obtained. Fig. 5.9 depicts a cross-sectional view of the topologically correct surfaces obtained with the ASTT filtering approach (red mesh) and the median filtering approach (blue mesh), superimposed on the segmented white matter volume. Note that the ASTT mesh follows the original surface more closely, especially along the sulci. Fig. 5.10 depicts the original surface, with the incorrect topology, and the topologically correct surface obtained with the ASTT filtering approach.

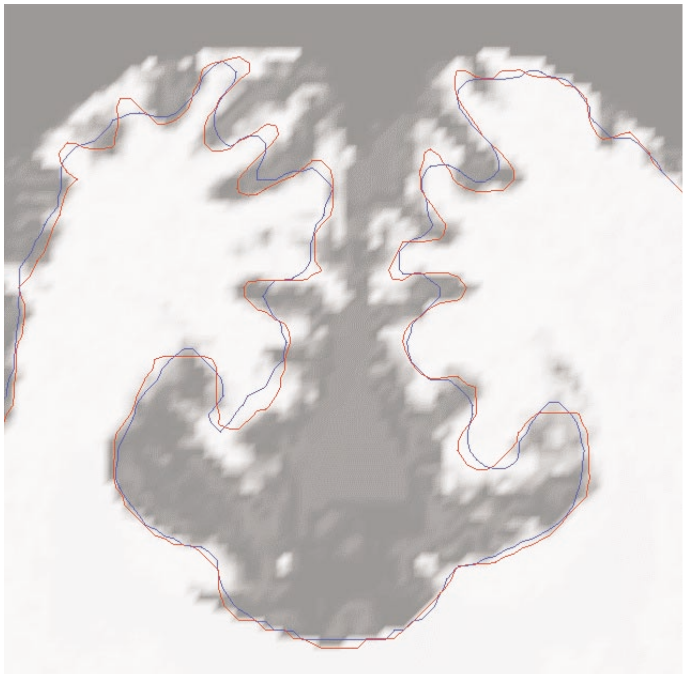


Figure 5.9: Cross-sectional view of the topologically correct surfaces obtained with the ASTT filtering approach (red mesh) and the median filtering approach (blue mesh), superimposed on the segmented white matter volume.

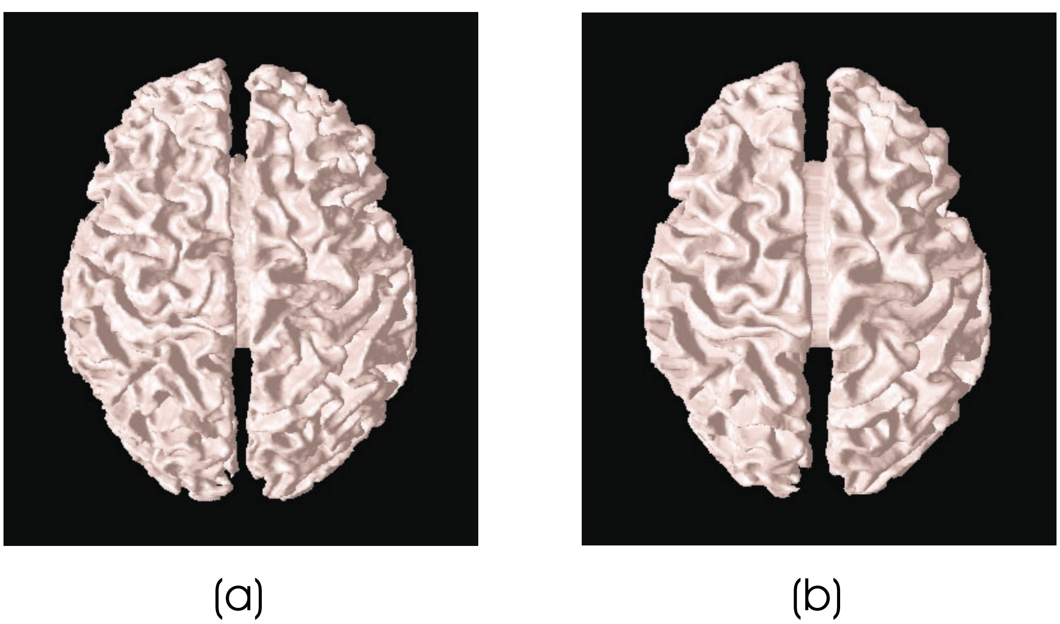


Figure 5.10: Rendering of surfaces. (a) Original (incorrect topology). (b) Result of ASTT filtering approach (correct topology).

Chapter 6

Multiscale Connectivity

The fundamental observation underlying the theory presented in this chapter is that it is natural to consider an object to be more or less connected than another object; i.e., a degree of connectivity is assigned to each object, which defines, equivalently, several levels of connectivity of varying strictness. We show that this leads to a novel theory of connectivity, to be referred to as *multiscale connectivity.*

The relevance of multiscale connectivity in image processing and analysis problems can be illustrated with the help of a simple figure. It is reasonable to assign an increasing degree of connectivity, from left to right, to the objects depicted in each row of Fig. 6.1. Equivalently, we may view the objects depicted in each row of Fig. 6.1 as manifestations of a unique object at different scales, with the scale decreasing from left to right (as remarked in Chapter 1, the term scale is used here in the sense of resolution, which is the inverse of the sense in which it is used, for example, in map making or in scale-space theory). Since objects at different scales have different degrees of connectivity, this leads naturally to the notion of multiscale connectivity.

This chapter is organized as follows. In Section 6.1, we deal with the axiomatic definition of continuous multiscale connectivity. We propose the equivalent notions of connectivity measure, which quantifies the degree of connectivity of an object, and connectivity pyramid, which provides several levels of connectivity, parameterized by scale. We also show that the fuzzy analogs of topological and graph-theoretic connectivity are examples of multiscale connectivity, and we introduce the notions of σ-connectivity openings and σ-reconstruction operators associated with a multiscale connectivity. In Section 6.2, we present the discrete version of the theory of multiscale connectivity. In Section 6.3, we study examples

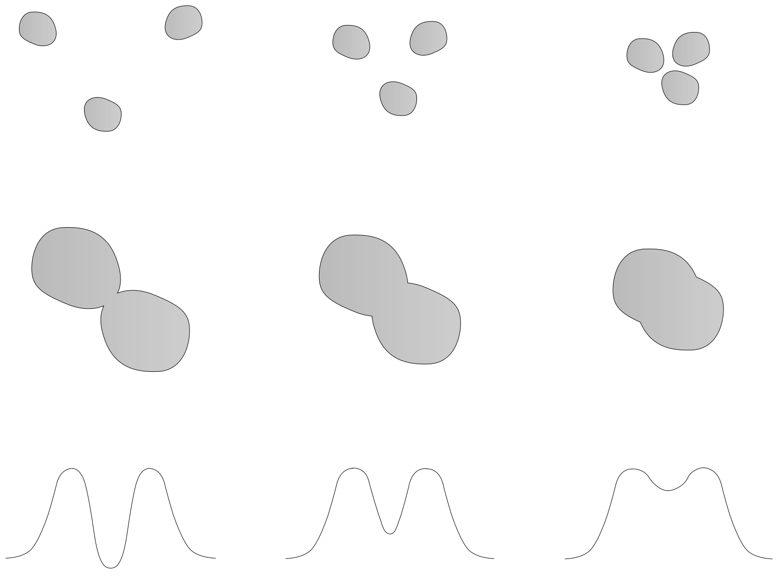


Figure 6.1: It is reasonable to assign an increasing degree of connectivity, from left to right, to the objects depicted in each row. These objects can alternatively be seen as manifestations of the same object at different scales, with the scale decreasing from left to right. This leads naturally to the notion of multiscale connectivity.

of multiscale connectivities generated by multiscale morphological operators, which include the cases of clustering pyramids and contraction pyramids. In Section 6.4, we study second-generation multiscale connectivities. In Section 6.5, we define a few useful multiscale tools based on multiscale connectivities, including pyramid decompositions, hierarchical segmentation, hierarchical clustering, and multiscale features, and present application examples using real discrete images. In Section 6.6, we investigate the notion of multiscale hyperconnectivity. Finally, in Section 6.7, we present a theory of multiscale connected operators.

6.1 Continuous Multiscale Connectivity

The idea of multiscale connectivity arises naturally from the observation that the connectivity of an object depends on the particular scale at which it is observed. Dependence of the notion of connectivity on scale can be characterized by either a measure of connectivity, which quantifies the degree of connectivity of an object, or by a connectivity class that depends on scale. In this section, we will see how these two approaches to multiscale connectivity can be axiomatized. Moreover, we will show that they are equivalent.

The following definition introduces the concept of a connectivity measure on a lattice $\mathcal{L}$.

6.1.1 Definition. Let $\mathcal{L}$ be a lattice with sup-generating family $\mathcal{S}$. A function φ: $\mathcal{L} \to \overline{\mathbb{R}}$ is said to be a *connectivity measure* on $\mathcal{L}$ if:

(i) $\varphi(O) = \varphi(x) = \infty$, for $x \in \mathcal{S}$,

(ii) for a family $\{A_\alpha\}$ in $\mathcal{L}$ such that $\bigwedge A_\alpha \neq O$, we have that $\varphi(\bigvee A_\alpha) \geq \bigwedge \varphi(A_\alpha)$. △

Given $A \in \mathcal{L}$, $\varphi(A)$ indicates the degree of connectivity of A. If $\varphi(A) = \infty$, A is said to be *fully connected*, whereas if $\varphi(A) = -\infty$, A is said to be *fully disconnected*. Intermediate connectivity, or σ*-connectivity*, is defined by saying that A is σ-connected if $\varphi(A) \geq \sigma$, for $\sigma \in \mathbb{R}$. Of course, if $\sigma \geq \tau$, then σ-connectivity implies τ-connectivity.

Axiom (i) of Definition 6.1.1 requires the zero element and the sup-generators to be fully connected. On the other hand, axiom (ii) requires that the degree of connectivity of the supremum of an "intersecting" family in $\mathcal{L}$ must not become smaller than the least degree of connectivity of the individual elements.

A connectivity measure φ is an $\overline{\mathbb{R}}$-fuzzy subset of $\mathcal{P}(\mathcal{L})$ (see Section 2.5 for basic definitions regarding fuzzy sets). In other words, a connectivity measure may be seen as a *fuzzy connectivity class*. We also remark that φ is a signed non-additive measure, also known in the literature as a *fuzzy measure* [28].

We define the σ*-sections* of a connectivity measure φ on $\mathcal{L}$ by $X_\sigma(\varphi) = \{A \in \mathcal{L} \mid \varphi(A) \geq \sigma\}$, for $\sigma \in \mathbb{R}$. Hence, the σ-section of φ contains the σ-connected elements of $\mathcal{L}$.

A connectivity measure φ on $\mathcal{L}$ is said to be *strong* if the greatest element I of $\mathcal{L}$ is fully connected; i.e., if $\varphi(I) = \infty$. In addition, if $\mathcal{L} = \mathcal{P}(E)$, with $E = \mathbb{R}^n$ or $E = \mathbb{Z}^n$, we say that φ is *translation-invariant* if $\varphi(A) = \varphi(A_h)$, for all $h \in E$.

Given a connectivity class $\mathcal{C}$ in $\mathcal{L}$, we can define a simple binary connectivity measure φ on $\mathcal{L}$, by letting $\varphi(A) = \infty$, if $A \in \mathcal{C}$, and $\varphi(A) = -\infty$, if $A \notin \mathcal{C}$. In other words, each $A \in \mathcal{L}$ is either fully connected, if $A \in \mathcal{C}$, or fully disconnected, if $A \notin \mathcal{C}$. Hence, connectivity classes correspond to *single-scale* connectivities, where the degree of connectivity is all-or-nothing; i.e., there is no intermediate connectivity.

6.1.2 Example. Consider the lattice $\mathcal{L} = \mathcal{P}(\mathbb{Z}^2)$ with the points as sup-generators. It can be easily verified that the function given by

$$\varphi(A) = \begin{cases} \infty, & \text{if } A \text{ is 4-adjacency connected} \\ 0, & \text{if } A \text{ is 8- but not 4-adjacency connected} \\ -\infty, & \text{otherwise} \end{cases}, \tag{6.1}$$

for $A \in \mathcal{L}$, defines a strong translation-invariant connectivity measure on $\mathcal{P}(\mathbb{Z}^2)$. ◊

The following result shows that the class of connectivity measures is closed under certain change-of-scale transformations.

6.1.3 Proposition. If φ is a connectivity measure on a lattice $\mathcal{L}$, and $f\colon \overline{\mathbb{R}} \to \overline{\mathbb{R}}$ is a nondecreasing and right-continuous function such that $f(\infty) = \infty$, then the composition $f(\varphi(\cdot))$ is a connectivity measure on $\mathcal{L}$. □

Proof. Let $\varphi' = f(\varphi(\cdot))$. From the assumption that $f(\infty) = \infty$, it follows that φ' satisfies axiom (*i*) of Definition 6.1.1. To show that axiom (*ii*) is satisfied as well, let $\{A_\alpha\}$ be a family in $\mathcal{L}$ such that $\bigwedge A_\alpha \neq O$. We have that $\varphi(\bigvee A_\alpha) \geq \bigwedge \varphi(A_\alpha)$, so that $\varphi'(\bigvee A_\alpha) = f(\varphi(\bigvee A_\alpha)) \geq f(\bigwedge \varphi(A_\alpha))$, since f is nondecreasing. But, it is easy to verify that every nondecreasing and right-continuous function f commutes with infimum; i.e., $f(\bigwedge x_\alpha) = \bigwedge f(x_\alpha)$, where $\{x_\alpha\} \subseteq \overline{\mathbb{R}}$. Hence, $\varphi'(\bigvee A_\alpha) \geq f(\bigwedge \varphi(A_\alpha)) = \bigwedge f(\varphi(A_\alpha)) = \bigwedge \varphi'(A_\alpha)$. Q.E.D.

Next, we give examples of application of the above proposition.

6.1.4 Example.

(a) If φ is a given connectivity measure on $\mathcal{L}$ and $a, b \in \mathbb{R}$, with $a > 0$, then $a\varphi + b$ is also a connectivity measure on $\mathcal{L}$, since the function $f(x) = ax + b$, $x \in \overline{\mathbb{R}}$, satisfies the requirements of Proposition 6.1.3. In other words, the class of connectivity measures is closed under *linear* change-of-scale transformations.

(b) Consider functions $f_\sigma, f^\sigma\colon \overline{\mathbb{R}} \to \overline{\mathbb{R}}$ given by

$$f_\sigma(x) = \begin{cases} -\infty, & \text{if } x < \sigma \\ x, & \text{if } x \geq \sigma \end{cases}, \quad f^\sigma(x) = \begin{cases} x, & \text{if } x < \sigma \\ \infty, & \text{if } x \geq \sigma \end{cases}, \tag{6.2}$$

where $\sigma \in \mathbb{R}$. It is easy to see that f_σ, f^σ satisfy the requirements of Proposition 6.1.3. Thus, given a connectivity measure φ on $\mathcal{L}$, we can define connectivity measures $\varphi_\sigma, \varphi^\sigma$ on $\mathcal{L}$ given by:

$$\varphi_\sigma(A) = \begin{cases} -\infty, & \text{if } \varphi(A) < \sigma \\ \varphi(A), & \text{if } \varphi(A) \geq \sigma \end{cases}, \quad \varphi^\sigma(A) = \begin{cases} \varphi(A), & \text{if } \varphi(A) < \sigma \\ \infty, & \text{if } \varphi(A) \geq \sigma \end{cases}, \tag{6.3}$$

for $A \in \mathcal{L}$. Note that φ_σ is the connectivity measure obtained by declaring the lattice elements with degree of connectivity less than σ to be fully disconnected, without affecting the other elements. On the other hand, φ^σ is such that the σ-connected lattice elements become fully connected, while all other elements are not affected. ◊

The next result shows that the class of connectivity measures is closed with respect to pointwise infimum.

6.1.5 Proposition. Let $\{\varphi_\alpha\}$ be an arbitrary family of connectivity measures on a lattice $\mathcal{L}$. The pointwise infimum $(\bigwedge \varphi_\alpha)(A) = \bigwedge \varphi_\alpha(A)$, for $A \in \mathcal{L}$, is a connectivity measure on $\mathcal{L}$. $\square$

Proof. Let $\varphi = \bigwedge \varphi_\alpha$. Clearly, φ satisfies axiom (i) of Definition 6.1.1. To show that axiom (ii) is satisfied as well, let $\{A_\beta\}$ be a family in $\mathcal{L}$ such that $\bigwedge A_\beta \neq O$. For each index α, we have that $\varphi_\alpha(\bigvee A_\beta) \geq \bigwedge_\beta \{\varphi_\alpha(A_\beta)\}$. Hence,

$$\varphi\left(\bigvee A_\beta\right) = \bigwedge_\alpha \varphi_\alpha\left(\bigvee A_\beta\right) \; \geq \bigwedge_\alpha \bigwedge_\beta \varphi_\alpha(A_\beta) = \bigwedge_\beta \bigwedge_\alpha \varphi_\alpha(A_\beta) = \bigwedge_\beta \varphi(A_\beta)\,, \tag{6.4}$$

which gives the desired result. Q.E.D.

The pointwise supremum of connectivity measures is not in general a connectivity measure. As a counterexample, let $\mathcal{L} = \mathcal{P}(\mathbb{R})$, with the points as sup-generators, and let $A = [0, 2]$ and $B = [1, 3]$. Take $\varphi_1(\emptyset) = \varphi_1(\{v\}) = \varphi_1(A) = \infty$, for $v \in \mathbb{R}$, while φ_1 takes on the value $-\infty$ everywhere else, and $\varphi_2(\emptyset) = \varphi_2(\{v\}) = \varphi_2(B) = \infty$, for $v \in \mathbb{R}$, while φ_2 takes on the value $-\infty$ everywhere else. It can be easily verified that φ_1 and φ_2 define connectivity measures on $\mathcal{P}(\mathbb{R})$ but $\varphi_1 \vee \varphi_2$ does not: we have $A \cap B \neq \emptyset$, but $(\varphi_1 \vee \varphi_2)(A \cup B) = -\infty \not\geq \min\{(\varphi_1 \vee \varphi_2)(A), (\varphi_1 \vee \varphi_2)(B)\} = \infty$, which contradicts axiom (ii) of connectivity measures.

Let $\mathcal{M}(\mathcal{L})$ be the set of all connectivity measures defined on a lattice $\mathcal{L}$, with a fixed sup-generating family. Note that $\mathcal{M}(\mathcal{L})$ is a poset under the product partial order $\varphi \leq \varphi'$ if $\varphi(A) \leq \varphi'(A)$, for all $A \in \mathcal{L}$. Moreover, we have the following result.

6.1.6 Proposition. Given a lattice $\mathcal{L}$, $\mathcal{M}(\mathcal{L})$ is a lattice, under the product partial order. $\square$

Proof. From Proposition 6.1.5, it is clear that $\mathcal{M}(\mathcal{L})$ is an inf semi-lattice under the product partial order. Moreover, the connectivity measure $\varphi(A) = \infty$, for all $A \in \mathcal{L}$, is the greatest element of $\mathcal{M}(\mathcal{L})$. From Proposition 2.1.1, it follows that $\mathcal{M}(\mathcal{L})$ is a complete lattice under the product partial order. Q.E.D.

A concept intimately related to connectivity measures is that of a connectivity pyramid. This is introduced by the following definition.

6.1.7 Definition. Let $\mathcal{L}$ be a lattice with sup-generating family $\mathcal{S}$. A *connectivity pyramid* on $\mathcal{L}$ is a mapping $\mathbf{C}$: $\mathbb{R} \to \mathcal{P}(\mathcal{L})$ such that:

(*i*) $\mathbf{C}(\sigma)$ is a connectivity class in $\mathcal{L}$, for each $\sigma \in \mathbb{R}$,

(*ii*) $\mathbf{C}(\sigma) \subseteq \mathbf{C}(\tau)$, if $\sigma \geq \tau$,

(*iii*) $\mathbf{C}(\sigma) = \bigcap_{\tau<\sigma} \mathbf{C}(\tau)$, for each $\sigma \in \mathbb{R}$. △

The connectivity class $\mathbf{C}(\sigma)$ is said to be the σ-*level* or the σ-*connectivity class* of $\mathbf{C}$, and it may be thought of as the connectivity class assigned at scale σ. For $A \in \mathcal{L}$, if A is connected at all scales, i.e., if $A \in \bigcap_{\sigma\in\mathbb{R}} \mathbf{C}(\sigma)$, then A is said to be fully connected, whereas if A is not connected at any scale, i.e., if $A \notin \bigcup_{\sigma\in\mathbb{R}} \mathbf{C}(\sigma)$, then A is said to be fully disconnected. Moreover, A is said to be σ-connected if $A \in \mathbf{C}(\sigma)$, for $\sigma \in \mathbb{R}$. Note that $\bigcap_{\sigma\in\mathbb{R}} \mathbf{C}(\sigma)$ is a connectivity class in $\mathcal{L}$ (since the set of all connectivity classes is closed under intersection – see Prop. 4.1.4(b)), so that the fully connected elements enjoy the same connectivity properties as the ones associated with a connectivity class (e.g., the supremum of overlapping fully connected elements is fully connected). However, the same does not apply to the fully disconnected elements.

Axiom (*ii*) of Definition 6.1.7 requires that the σ-levels of a connectivity pyramid be nested, so that fewer elements are connected as one moves upward in the pyramid (i.e., a connected element at a given level of the pyramid may not be connected at a higher level). In other words, more objects tend to be connected at small scales than at large scales. On the other hand, the semi-continuity axiom (*iii*) provides a smoothness constraint on the levels of a connectivity pyramid. Note that axiom (*iii*) actually implies axiom (*ii*).

A connectivity pyramid $\mathbf{C}$ on $\mathcal{L}$ is said to be *strong* if each σ-connectivity class is strong; i.e., if $I \in \mathbf{C}(\sigma)$, for all $\sigma \in \mathbb{R}$ (so that I is fully connected). In addition, if $\mathcal{L} = \mathcal{P}(E)$, with $E = \mathbb{R}^n$ or $E = \mathbb{Z}^n$, we say that $\mathbf{C}$ is *translation-invariant* if each σ-connectivity class is translation-invariant; i.e., if $A \in \mathbf{C}(\sigma) \Leftrightarrow A_h \in \mathbf{C}(\sigma)$, for all $h \in E$, $\sigma \in \mathbb{R}$.

Given a connectivity class $\mathcal{C}$ in $\mathcal{L}$, we can define a simple connectivity pyramid $\mathbf{C}$ on $\mathcal{L}$, by setting $\mathbf{C}(\sigma) = \mathcal{C}$, for all $\sigma \in \mathbb{R}$. In this case, each $A \in \mathcal{L}$ is either fully connected, if $A \in \mathcal{C}$, or fully disconnected, if $A \notin \mathcal{C}$. Hence, connectivity classes correspond to single-scale connectivities, where the connectivity is the same at all scales.

Next, we give a characterization of the set $\mathcal{Y}(\mathcal{L})$ of all connectivity pyramids in a lattice $\mathcal{L}$, with a fixed sup-generating family. Let us introduce the following sets:

$$\mathcal{Y}^0(\mathcal{L}) = \text{Fun}(\mathbb{R}, \mathcal{P}(\mathcal{L})), \tag{6.5}$$

$$\mathcal{Y}^1(\mathcal{L}) = \{\mathbf{F} \in \mathcal{Y}^0(\mathcal{L}) \mid \mathbf{F} \text{ satisfies axiom } (i) \text{ of a connectivity pyramid}\}, \tag{6.6}$$

$$\mathcal{Y}^2(\mathcal{L}) = \{\mathbf{F} \in \mathcal{Y}^0(\mathcal{L}) \mid \mathbf{F} \text{ satisfies axioms } (i) \text{ and } (ii) \text{ of a connectivity pyramid}\}. \tag{6.7}$$

Note that $\mathcal{Y}(\mathcal{L}) \subseteq \mathcal{Y}^2(\mathcal{L}) \subseteq \mathcal{Y}^1(\mathcal{L}) \subseteq \mathcal{Y}^0(\mathcal{L})$. Note also that $\mathcal{Y}^0(\mathcal{L})$ is a lattice, under the product inclusion order $\mathbf{F} \leq \mathbf{F}'$ if $\mathbf{F}(\sigma) \subseteq \mathbf{F}'(\sigma)$, for $\sigma \in \mathbb{R}$, with supremum and infimum given by the pointwise union $(\bigvee \mathbf{F}_\alpha)(\sigma) = \bigcup \mathbf{F}_\alpha(\sigma)$, for $\sigma \in \mathbb{R}$, and the pointwise intersection $(\bigwedge \mathbf{F}_\alpha)(\sigma) = \bigcap \mathbf{F}_\alpha(\sigma)$, for $\sigma \in \mathbb{R}$, respectively.

6.1.8 Proposition. Let $\mathcal{L}$ be a lattice, and let $\bigvee$ and $\bigwedge$ denote the supremum and infimum operations in lattice $\mathcal{Y}^0(\mathcal{L})$, respectively.

(a) The operator Φ on $\mathcal{Y}^0(\mathcal{L})$ given by

$$\Phi(\mathbf{F})(\sigma) = \phi(\mathbf{F}(\sigma)), \quad \sigma \in \mathbb{R}, \tag{6.8}$$

for $\mathbf{F} \in \mathcal{Y}^0(\mathcal{L})$, with ϕ as in (4.7), is a closing on $\mathcal{Y}^0(\mathcal{L})$, with invariance domain $\text{Inv}(\Phi) = \mathcal{Y}^1(\mathcal{L})$.

(b) $\mathcal{Y}^1(\mathcal{L})$ is an underlattice of $\mathcal{Y}^0(\mathcal{L})$, with infimum $\bigwedge \mathbf{F}_\alpha$ and supremum $\Phi(\bigvee \mathbf{F}_\alpha)$.

(c) $\mathcal{Y}^2(\mathcal{L})$ is a sublattice of $\mathcal{Y}^1(\mathcal{L})$, and therefore an underlattice of $\mathcal{Y}^0(\mathcal{L})$, with infimum $\bigwedge \mathbf{F}_\alpha$ and supremum $\Phi(\bigvee \mathbf{F}_\alpha)$.

(d) The operator Ω on $\mathcal{Y}^1(\mathcal{L})$ given by

$$\Omega(\mathbf{F})(\sigma) = \bigcap_{\tau < \sigma} \mathbf{F}(\tau), \quad \sigma \in \mathbb{R}, \tag{6.9}$$

for $\mathbf{F} \in \mathcal{Y}^1(\mathcal{L})$, is a morphological filter on $\mathcal{Y}^1(\mathcal{L})$, while the restriction Ω' of Ω to $\mathcal{Y}^2(\mathcal{L})$ is a closing on $\mathcal{Y}^2(\mathcal{L})$. Moreover, we have that $\text{Inv}(\Omega) = \text{Inv}(\Omega') = \mathcal{Y}(\mathcal{L})$.

(e) $\mathcal{Y}(\mathcal{L})$ is an underlattice of both $\mathcal{Y}^1(\mathcal{L})$ and $\mathcal{Y}^2(\mathcal{L})$, and therefore of $\mathcal{Y}^0(\mathcal{L})$, with infimum $\bigwedge \mathbf{C}_\alpha$ and supremum $\Omega\,\Phi(\bigvee \mathbf{C}_\alpha)$. □

Proof. (a): That Φ is a closing on $\mathcal{Y}^0(\mathcal{L})$ follows directly from the fact that ϕ is a closing on $\mathcal{P}(\mathcal{L})$. Now, $\Phi(\mathbf{F}) = \mathbf{F} \;\Leftrightarrow\; \phi(\mathbf{F}(\sigma)) = \mathbf{F}(\sigma)$, for all $\sigma \in \mathbb{R} \;\Leftrightarrow\; \mathbf{F}(\sigma) \in \mathrm{Inv}(\phi) = \mathrm{Ccl}(\mathcal{L})$, for all $\sigma \in \mathbb{R} \;\Leftrightarrow\; \mathbf{F} \in \mathcal{Y}^1(\mathcal{L})$. Hence, $\mathrm{Inv}(\Phi) = \mathcal{Y}^1(\mathcal{L})$.

(b): This follows directly from part (a) and Proposition 2.2.2(a).

(c): The proof of this is obvious.

(d): Clearly, Ω is increasing. To show that Ω is idempotent, note that, for each $\sigma \in \mathbb{R}$, we have that $\Omega\Omega(\mathbf{F})(\sigma) = \bigcap_{\tau<\sigma} \Omega(\mathbf{F})(\tau) = \bigcap_{\tau<\sigma} \bigcap_{\tau'<\tau} \mathbf{F}(\tau')$. Now, it is easy to verify that $\bigcap_{b<a} \bigcap_{c<b} A_c = \bigcap_{b<a} A_b$, for any indexed family of sets $\{A_a \mid a \in \mathbb{R}\}$. Hence, $\Omega\Omega(\mathbf{F})(\sigma) = \bigcap_{\tau<\sigma} \mathbf{F}(\tau) = \Omega(\mathbf{F})(\sigma)$, for all $\sigma \in \mathbb{R}$; i.e., $\Omega\Omega = \Omega$, as required. This shows that Ω is a morphological filter on $\mathcal{Y}^1(\mathcal{L})$. To show that the restriction Ω' is a closing on $\mathcal{Y}^2(\mathcal{L})$, we need to show that Ω is extensive on $\mathcal{Y}^2(\mathcal{L})$. For any $\mathbf{F} \in \mathcal{Y}^2(\mathcal{L})$, we have that $\mathbf{F}(\tau) \supseteq \mathbf{F}(\sigma)$, for all $\tau < \sigma$, so that $\Omega(\mathbf{F})(\sigma) = \bigcap_{\tau<\sigma} \mathbf{F}(\tau) \supseteq \mathbf{F}(\sigma)$, for all $\sigma \in \mathbb{R}$; i.e., $\Omega(\mathbf{F}) \geq \mathbf{F}$, as required. Now, $\mathbf{F} \in \mathcal{Y}^1(\mathcal{L})$ or $\mathbf{F} \in \mathcal{Y}^2(\mathcal{L})$ and $\Omega(\mathbf{F}) = \mathbf{F}$ imply that $\mathbf{F}$ satisfies all three axioms of a connectivity pyramid, so that $\mathbf{F} \in \mathcal{Y}(\mathcal{L})$. Conversely, if $\mathbf{C} \in \mathcal{Y}(\mathcal{L})$ then, by axiom *(iii)* of a connectivity pyramid, we have that $\Omega(\mathbf{C})(\sigma) = \bigcap_{\tau<\sigma} \mathbf{C}(\tau) = \mathbf{C}(\sigma)$, for all $\sigma \in \mathbb{R}$, so that $\Omega(\mathbf{C}) = \mathbf{C}$. Hence, $\mathrm{Inv}(\Omega) = \mathrm{Inv}(\Omega') = \mathcal{Y}(\mathcal{L})$.

(e) From part (d) and Proposition 2.2.2(a), $\mathcal{Y}(\mathcal{L})$ is an underlattice of $\mathcal{Y}^2(\mathcal{L})$, with the same infimum $\bigwedge \mathbf{C}_\alpha$, and supremum $\Omega'(\bigvee^2 \mathbf{C}_\alpha) = \Omega(\bigvee^2 \mathbf{C}_\alpha)$, where $\bigvee^2$ denotes the supremum in $\mathcal{Y}^2(\mathcal{L})$. But, since $\mathcal{Y}^2(\mathcal{L})$ is a sublattice of $\mathcal{Y}^1(\mathcal{L})$, it follows that $\mathcal{Y}(\mathcal{L})$ is also an underlattice of $\mathcal{Y}^1(\mathcal{L})$, with the same infimum $\bigwedge \mathbf{C}_\alpha$ and supremum $\Omega(\bigvee^1 \mathbf{C}_\alpha)$, where $\bigvee^1$ is the supremum in $\mathcal{Y}^1(\mathcal{L})$. Now, from part (b), we have that $\bigvee^1 \mathbf{F}_\alpha = \Phi(\bigvee \mathbf{F}_\alpha)$; hence, the supremum in $\mathcal{Y}(\mathcal{L})$ is given by $\Omega\,\Phi(\bigvee \mathbf{C}_\alpha)$. Q.E.D.

The previous result has some important consequences:

- The sets $\mathcal{Y}^0(\mathcal{L})$, $\mathcal{Y}^1(\mathcal{L})$, $\mathcal{Y}^2(\mathcal{L})$, and $\mathcal{Y}(\mathcal{L})$ are complete lattices under the product inclusion order.
- The lattices $\mathcal{Y}^0(\mathcal{L})$, $\mathcal{Y}^1(\mathcal{L})$, $\mathcal{Y}^2(\mathcal{L})$, and $\mathcal{Y}(\mathcal{L})$ share the same infimum, namely the pointwise intersection.
- Since, for an idempotent operator ψ, $\mathrm{Inv}(\psi) = \mathrm{Range}(\psi)$, it follows that $\mathcal{Y}^1(\mathcal{L}) = \mathrm{Range}(\Phi)$ and $\mathcal{Y}(\mathcal{L}) = \mathrm{Range}(\Omega) = \mathrm{Range}(\Omega')$.

In particular, we derive the following conclusions about the set $\mathcal{Y}(\mathcal{L})$ of connectivity pyramids. The set $\mathcal{Y}(\mathcal{L})$ is a complete lattice under the product inclusion order $\mathbf{C} \leq \mathbf{C}'$ if

$\mathbf{C}(\sigma) \subseteq \mathbf{C}'(\sigma)$, for all $\sigma \in \mathbb{R}$. The infimum in $\mathcal{Y}(\mathcal{L})$ is given by pointwise intersection. It follows that, given an arbitrary family $\{\mathbf{C}_\alpha\}$ of connectivity pyramids on $\mathcal{L}$, then $\mathbf{C}$, given by $\mathbf{C}(\sigma) = \bigcap \mathbf{C}_\alpha(\sigma)$, for all $\sigma \in \mathbb{R}$, is a connectivity pyramid on $\mathcal{L}$. Moreover, given an $\mathbf{F}$ such that $\mathbf{F} \in \mathcal{Y}^1(\mathcal{L})$ or $\mathbf{F} \in \mathcal{Y}^2(\mathcal{L})$, one obtains a connectivity pyramid $\mathbf{C} = \Omega(\mathbf{F})$ by applying Ω to $\mathbf{F}$. In addition, given an $\mathbf{F} \in \mathcal{Y}^0(\mathcal{L})$, one obtains a connectivity pyramid $\mathbf{C} = \Omega\,\Phi(\mathbf{F})$ by applying Φ, and then Ω, to $\mathbf{F}$.

Connectivity pyramids are closely related to connectivity measures. Given a connectivity measure on $\mathcal{L}$, one can define a unique connectivity pyramid on $\mathcal{L}$, and vice-versa. In addition, this bijection is order-preserving. This is shown by the following theorem.

6.1.9 Theorem. Let $\mathcal{L}$ be a lattice with sup-generating family $\mathcal{S}$. The lattice $\mathcal{M}(\mathcal{L})$ of connectivity measures on $\mathcal{L}$ is isomorphic to the lattice $\mathcal{Y}(\mathcal{L})$ of connectivity pyramids on $\mathcal{L}$. Moreover, the isomorphism Γ: $\mathcal{M}(\mathcal{L}) \to \mathcal{Y}(\mathcal{L})$ is given by

$$\Gamma(\varphi)(\sigma) = \{A \in \mathcal{L} \mid \varphi(A) \geq \sigma\}, \quad \sigma \in \mathbb{R}, \tag{6.10}$$

with inverse Γ^{-1}: $\mathcal{Y}(\mathcal{L}) \to \mathcal{M}(\mathcal{L})$, given by

$$\Gamma^{-1}(\mathbf{C})(A) = \bigvee \{\sigma \in \mathbb{R} \mid A \in \mathbf{C}(\sigma)\}, \quad A \in \mathcal{L}. \tag{6.11}$$

□

Proof. First, we show that Γ is a mapping from $\mathcal{M}(\mathcal{L})$ into $\mathcal{Y}(\mathcal{L})$; i.e., we show that $\mathbf{C} = \Gamma(\varphi)$, where $\varphi \in \mathcal{M}(\mathcal{L})$, defines a connectivity pyramid on $\mathcal{L}$. Note that $\varphi(O) = \varphi(x) = \infty$, for all $x \in \mathcal{S}$, implies that $O \in \mathbf{C}(\sigma)$ and $\mathcal{S} \subseteq \mathbf{C}(\sigma)$, for all $\sigma \in \mathbb{R}$. Now, consider a family $\{A_\alpha\}$ in $\mathbf{C}(\sigma)$, such that $\bigwedge A_\alpha \neq O$. Note that $\varphi(A_\alpha) \geq \sigma$, for each index α. Since φ is a connectivity measure, we have that $\varphi(\bigvee A_\alpha) \geq \bigwedge \varphi(A_\alpha) \geq \sigma \Rightarrow \bigvee A_\alpha \in \mathbf{C}(\sigma)$. Hence, $\mathbf{C}(\sigma)$ is a connectivity class, for all $\sigma \in \mathbb{R}$, which shows axiom (i) of Definition 6.1.7. Axiom (ii) follows easily from (6.10), while axiom (iii) follows from the fact that $A \in \mathbf{C}(\sigma) \Leftrightarrow \varphi(A) \geq \sigma \Leftrightarrow \varphi(A) \geq \tau, \forall\, \tau < \sigma \Leftrightarrow A \in \mathbf{C}(\tau), \forall\, \tau < \sigma \Leftrightarrow A \in \bigcap_{\tau<\sigma} \mathbf{C}(\tau)$, for each $\sigma \in \mathbb{R}$.

Now, we show that Γ^{-1} defines a mapping from $\mathcal{Y}(\mathcal{L})$ into $\mathcal{M}(\mathcal{L})$; i.e., we show that $\varphi = \Gamma^{-1}(\mathbf{C})$, where $\mathbf{C} \in \mathcal{Y}(\mathcal{L})$, defines a connectivity measure on $\mathcal{L}$. Note that $O \in \mathbf{C}(\sigma)$ and $\mathcal{S} \subseteq \mathbf{C}(\sigma)$, for all $\sigma \in \mathbb{R}$, implies that $\varphi(O) = \varphi(x) = \bigvee \mathbb{R} = \infty$, for all $x \in \mathcal{S}$, which shows axiom (i) of Definition 6.1.1. To show axiom (ii), note that, since $\mathbf{C}(\cdot)$ is decreasing, we have that $\varphi(A) \geq \sigma \Leftrightarrow A \in \mathbf{C}(\tau), \forall\, \tau < \sigma$, for all $\sigma \in \mathbb{R} \cup \{\infty\}$. Consider a family $\{A_\alpha\}$ in $\mathcal{L}$ such that $\bigwedge A_\alpha \neq O$, and let $\sigma = \bigwedge \varphi(A_\alpha)$. If $\sigma = -\infty$, obviously $\varphi(\bigvee A_\alpha) \geq \sigma$, and we are done. Otherwise, for each index α, $\varphi(A_\alpha) \geq \sigma \Rightarrow A_\alpha \in \mathbf{C}(\tau), \forall\, \tau < \sigma \Rightarrow \bigvee A_\alpha \in \mathbf{C}(\tau), \forall\, \tau < \sigma$, which implies that $\varphi(\bigvee A_\alpha) \geq \sigma$, as required.

Now, we show that the lattices $\mathcal{M}(\mathcal{L})$ and $\mathcal{Y}(\mathcal{L})$ are isomorphic. The mappings Γ and Γ^{-1} are clearly order-preserving. It therefore remains to be shown that Γ and Γ^{-1} are the inverses of each other. Let $\varphi \in \mathcal{M}(\mathcal{L})$. We have that $\Gamma^{-1}\Gamma(\varphi)(A) = \bigvee\{\sigma \in \mathbb{R} \mid A \in \Gamma(\varphi)(\sigma)\} = \bigvee\{\sigma \in \mathbb{R} \mid \varphi(A) \geq \sigma\} = \varphi(A)$, for all $A \in \mathcal{L}$. Hence, $\Gamma^{-1}\Gamma(\varphi) = \varphi$. Now, let $\mathbf{C} \in \mathcal{Y}(\mathcal{L})$. First, we show that $\Gamma^{-1}(\mathbf{C})(A) \geq \sigma \;\Leftrightarrow\; A \in \mathbf{C}(\sigma)$, for $\sigma \in \mathbb{R}$. The reverse implication is obvious. To show the direct implication, suppose that $A \notin \mathbf{C}(\sigma) = \bigcap_{\tau<\sigma} \mathbf{C}(\tau)$. This implies that $A \notin \mathbf{C}(\tau)$, for some $\tau < \sigma$, so that $\Gamma^{-1}(\mathbf{C})(A) \not\geq \sigma$, establishing the desired result. It follows that $\Gamma\Gamma^{-1}(\mathbf{C})(\sigma) = \{A \in \mathcal{L} \mid \Gamma^{-1}(\mathbf{C})(A) \geq \sigma\} = \mathbf{C}(\sigma)$, for all $\sigma \in \mathbb{R}$. Hence, $\Gamma\Gamma^{-1}(\mathbf{C}) = \mathbf{C}$. Q.E.D.

The isomorphism between lattices $\mathcal{M}(\mathcal{L})$ and $\mathcal{Y}(\mathcal{L})$ is of course a bijection; i.e., to each connectivity measure φ on $\mathcal{L}$, there is an associated equivalent connectivity pyramid $\mathbf{C}$ on $\mathcal{L}$, which consists of the σ-sections of φ. Conversely, φ can be regenerated by "stacking up" the σ-levels of $\mathbf{C}$. Hence, a multiscale connectivity on $\mathcal{L}$ can be equivalently specified by either method. Depending on the circumstances, one method can be more convenient than the other. Therefore, we often say that $\mathcal{L}$ is furnished with a *multiscale connectivity system* $(\varphi, \mathbf{C}) \in \mathcal{M}(\mathcal{L}) \times \mathcal{Y}(\mathcal{L})$, such that φ and $\mathbf{C}$ are equivalent under the bijection given in Theorem 6.1.9.

Note that the previous definitions regarding σ-connectivity, given by means of connectivity measures and connectivity pyramids, agree with each other. For example, we have that $\varphi(A) \geq \sigma \;\Leftrightarrow\; A \in \mathbf{C}(\sigma)$, in which case A is σ-connected, for $\sigma \in \mathbb{R}$. In addition, $\varphi(A) = \infty \;\Leftrightarrow\; A \in \bigcap_{\sigma\in\mathbb{R}} \mathbf{C}(\sigma)$, in which case A is fully connected. Similarly, $\varphi(A) = -\infty \;\Leftrightarrow\; A \notin \bigcup_{\sigma\in\mathbb{R}} \mathbf{C}(\sigma)$, in which case A is fully disconnected. Moreover, we have that φ is strong (resp. translation-invariant) if and only if $\mathbf{C}$ is strong (resp. translation-invariant), in which case $(\varphi, \mathbf{C})$ is said to be a strong (resp. translation-invariant) multiscale connectivity system.

Clearly, the pair (Γ, Γ^{-1}) of operators defined in Theorem 6.1.9 is an adjunction between $\mathcal{M}(\mathcal{L})$ and $\mathcal{Y}(\mathcal{L})$. The next result shows that (Γ, Γ^{-1}) is an adjunction between $\mathcal{M}(\mathcal{L})$ and $\mathcal{Y}^2(\mathcal{L})$ as well. Furthermore, the resulting closing $\Gamma\Gamma^{-1}$ on $\mathcal{Y}^2(\mathcal{L})$ is equal to the closing Ω' of Proposition 6.1.8(d).

6.1.10 Proposition. Let Γ and Γ^{-1} be defined as in (6.10) and (6.11), respectively. The pair (Γ, Γ^{-1}) defines an adjunction between $\mathcal{M}(\mathcal{L})$ and $\mathcal{Y}^2(\mathcal{L})$. In addition, $\Gamma\Gamma^{-1} = \Omega'$, where Ω' is the closing on $\mathcal{Y}^2(\mathcal{L})$ defined in Proposition 6.1.8(d). □

Proof. First, we show that Γ^{-1} defines an operator from $\mathcal{Y}^2(\mathcal{L})$ into $\mathcal{M}(\mathcal{L})$; i.e., we show that $\varphi = \Gamma^{-1}(\mathbf{F})$, where $\mathbf{F} \in \mathcal{Y}^2(\mathcal{L})$, defines a connectivity measure on $\mathcal{L}$. The proof is essentially the same as the corresponding step in the proof of Theorem 6.1.9. Note that $\varphi(A) = \bigvee\{\sigma \in \mathbb{R} \mid A \in \mathbf{F}(\sigma)\}$, for $A \in \mathcal{L}$. We have that $O \in \mathbf{F}(\sigma)$ and $\mathcal{S} \subseteq \mathbf{F}(\sigma)$, for all $\sigma \in \mathbb{R}$; hence, $\varphi(O) = \varphi(x) = \bigvee \mathbb{R} = \infty$, for all $x \in \mathcal{S}$, which shows axiom (i) of Definition 6.1.7. To show axiom (ii), note that, since $\mathbf{F}(\sigma)$ is decreasing, we have that $\varphi(A) \geq \sigma \;\Leftrightarrow\; A \in \mathbf{F}(\tau), \forall \tau < \sigma$, for all $\sigma \in \mathbb{R} \cup \{\infty\}$. Consider a family $\{A_\alpha\}$ in $\mathcal{L}$ such that $\bigwedge A_\alpha \neq O$, and let $\sigma = \bigwedge \phi(A_\alpha)$. If $\sigma = -\infty$, then $\phi(\bigvee A_\alpha) \geq \sigma$ and we are done. Otherwise, for each index α, $\phi(A_\alpha) \geq \sigma \Rightarrow A_\alpha \in \mathbf{F}(\tau)$, $\forall \tau < \sigma \Rightarrow \bigvee A_\alpha \in \mathbf{F}(\tau)$, $\forall \tau < \sigma$, which implies that $\phi(\bigvee A_\alpha) \geq \sigma$, as required.

Now, we show that (Γ, Γ^{-1}) defines an adjunction between $\mathcal{M}(\mathcal{L})$ and $\mathcal{Y}^2(\mathcal{L})$; i.e., for $\varphi \in \mathcal{M}(\mathcal{L})$ and $\mathbf{F} \in \mathcal{Y}^2(\mathcal{L})$, we have that $\Gamma^{-1}(\mathbf{F}) \leq \varphi \;\Leftrightarrow\; \mathbf{F} \leq \mathbf{C}$, where $\mathbf{C} = \Gamma(\varphi)$ is the connectivity pyramid associated with the connectivity measure φ via Γ. Assume that $\Gamma^{-1}(\mathbf{F}) \leq \varphi$. For all $A \in \mathbf{F}(\sigma)$, we have that $\sigma \leq \Gamma^{-1}(\mathbf{F})(A) \leq \varphi(A)$. But $\varphi(A) \geq \sigma$ implies that $A \in \mathbf{C}(\sigma)$. Since this is true for all $\sigma \in \mathbb{R}$, we have that $\mathbf{F} \leq \mathbf{C}$. Conversely, assume that $\mathbf{F} \leq \mathbf{C}$. Clearly, we have that $\{\sigma \in \mathbb{R} \mid A \in \mathbf{F}(\sigma)\} \subseteq \{\sigma \in \mathbb{R} \mid A \in \mathbf{C}(\sigma)\}$. Hence, $\Gamma^{-1}(\mathbf{F}) = \bigvee\{\sigma \in \mathbb{R} \mid A \in \mathbf{F}(\sigma)\} \leq \bigvee\{\sigma \in \mathbb{R} \mid A \in \mathbf{C}(\sigma)\} = \Gamma^{-1}(\mathbf{C}) = \varphi$.

Finally, we show that $\Gamma\Gamma^{-1} = \Omega'$ on $\mathcal{Y}^2(\mathcal{L})$. By using the fact that two closings are equal if and only if they have the same domain of invariance (see Proposition 2.2.1(b)), and Proposition 6.1.8(d), it is sufficient to show that $\mathrm{Inv}(\Gamma\Gamma^{-1}) = \mathcal{Y}(\mathcal{L})$. Let $\mathbf{F} \in \mathcal{Y}^2(\mathcal{L})$. We have shown above that $\Gamma^{-1}(\mathbf{F}) \in \mathcal{M}(\mathcal{L})$, which implies that $\Gamma\Gamma^{-1}(\mathbf{F}) \in \mathcal{Y}(\mathcal{L})$. Therefore, $\Gamma\Gamma^{-1}(\mathbf{F}) = \mathbf{F} \Rightarrow \mathbf{F} \in \mathcal{Y}(\mathcal{L})$. Conversely, let $\mathbf{C} \in \mathcal{Y}(\mathcal{L})$. Since Γ is the inverse of Γ^{-1}, we get that $\Gamma\Gamma^{-1}(\mathbf{C}) = \mathbf{C}$; i.e., $\mathbf{C} \in \mathrm{Inv}(\Gamma\Gamma^{-1})$. Q.E.D.

The previous result implies that, given $\mathbf{F} \in \mathcal{Y}^2(\mathcal{L})$, one obtains a multiscale connectivity system $(\varphi, \mathbf{C})$ on $\mathcal{L}$ by setting $\varphi = \Gamma^{-1}(\mathbf{F})$ and $\mathbf{C} = \Omega'(\mathbf{F})$.

Note that, given an $\mathbf{F} \in \mathcal{Y}^1(\mathcal{L})$, one *does not* in general obtain a connectivity measure by applying Γ^{-1} to $\mathbf{F}$ (even though, as remarked previously, one does obtain a connectivity pyramid $\mathbf{C} = \Omega(\mathbf{F})$ by applying Ω to $\mathbf{F}$). As a counterexample, let $\mathcal{L} = \mathcal{P}(\mathbb{R})$, with the points as sup-generators. Let $\mathcal{C}_1$ be the connectivity class in $\mathcal{P}(\mathbb{R})$ containing the empty set, the points, and set $C = [0, 2]$. Similarly, let $\mathcal{C}_2$ be the connectivity class in $\mathcal{P}(\mathbb{R})$ containing the empty set, the points, and set $D = [1, 3]$. Define $\mathbf{F} \in \mathcal{Y}^1(\mathcal{L})$ by setting $\mathbf{F}(\sigma) = \mathcal{C}_1$, if σ is rational, and $\mathbf{F}(\sigma) = \mathcal{C}_2$, otherwise. Let $\xi = \Gamma^{-1}(\mathbf{F})$; i.e., $\xi(A) = \bigvee\{\sigma \in \mathbb{R} \mid A \in \mathbf{F}(\sigma)\}$, for $A \in \mathcal{P}(\mathbb{R})$. Clearly, $\xi(C) = \xi(D) = \infty$, but $\xi(C \cup D) = -\infty \ngeq \min\{\xi(C), \xi(D)\} = \infty$, even though $C \cap D \neq \emptyset$, which contradicts axiom (*iii*) of Definition 6.1.1. Therefore, ξ is not a connectivity measure.

Recall from Sections 3.3 and 3.4 the notions of fuzzy topological τ-connectivity and fuzzy graph-theoretic τ-connectivity, respectively. We show next that these notions give rise to multiscale connectivities, which provide multiscale extensions of the classical notions of topological and graph-theoretic connectivity, respectively. In the following result, recall from Section 2.5 the concept of a fuzzy topological space generated by a topology pyramid.

6.1.11 Proposition. (*Multiscale Topological Connectivity*). Let $\mathcal{L} = \mathcal{P}(E)$ with the points as sup-generators. Let $\mathbf{P} = \{\mathcal{G}_\tau \mid \tau \in \overline{\mathbb{R}} \setminus \{\infty\}\}$ be a topology pyramid on E, and let $(E, \Delta(\mathbf{P}))$ be the $\overline{\mathbb{R}}$-fuzzy topological space generated by $\mathbf{P}$. Then, $\mathbf{C}\colon \mathbb{R} \to \mathcal{P}(\mathcal{P}(E))$, given by

$$\mathbf{C}(\tau) = \{A \subseteq E \mid A \text{ is } \tau\text{-connected in } (E, \Delta(\mathbf{P}))\}, \quad \tau \in \mathbb{R}, \tag{6.12}$$

defines a connectivity pyramid on $\mathcal{P}(E)$. □

PROOF. From Proposition 3.3.5, we have that $\mathbf{C}(\tau) = \{A \subseteq E \mid A \text{ is connected in } (E, \mathcal{G}_{-\tau})\}$, for $\tau \in \mathbb{R}$. It follows immediately that $\mathbf{C}(\tau)$ is a connectivity class, for all $\tau \in \mathbb{R}$, which shows axiom (*i*) of a connectivity pyramid. From the fact that $\mathcal{G}_{-\tau_2} \subseteq \mathcal{G}_{-\tau_1}$, for $\tau_2 \leq \tau_1$, and the observation that a connected set, according to a topology $\mathcal{G}_1$, is connected in a coarser topology $\mathcal{G}_2 \subseteq \mathcal{G}_1$, we conclude that $\mathbf{C}(\cdot)$ is a decreasing mapping, which shows axiom (*ii*) of a connectivity pyramid. We now show axiom (*iii*). Given $\tau \in \mathbb{R}$, the inclusion $\mathbf{C}(\tau) \subseteq \bigcap_{s<\tau} \mathbf{C}(s)$ follows directly from the fact that $\mathbf{C}(\cdot)$ is decreasing. We show the converse inclusion by establishing the contrapositive. Suppose that $A \notin \mathbf{C}(\tau)$; i.e., there is a separation (G_1, G_2) of A in $\mathcal{G}_{-\tau}$. Since $\mathbf{P}$ is a topology pyramid, we have that $\mathcal{G}_{-\tau} = \bigcup_{s>-\tau} \mathcal{G}_s$, which implies that there is a $\tau_0 > -\tau$ such that $G_1, G_2 \in \mathcal{G}_{\tau_0}$, so that $A \notin \mathbf{C}(-\tau_0)$. In other words, there is an $s < \tau$, namely $s = -\tau_0$, such that $A \notin \mathbf{C}(s)$. But this implies that $A \notin \bigcap_{s<\tau} \mathbf{C}(s)$, as required. Q.E.D.

Therefore, in a multiscale topological connectivity framework, the connected sets at scale τ correspond to the τ-connected sets in $(E, \Delta(\mathbf{P}))$, which in turn correspond to the connected sets in the topological space $(E, \mathcal{G}_{-\tau})$, for $\tau \in \mathbb{R}$ (so that the τ-level topology in $\mathbf{P}$ defines connectivity at scale $-\tau$). Furthermore, it is easy to verify that A is fully connected, i.e., $A \in \bigcap_{\tau \in \mathbb{R}} \mathbf{C}(\tau)$, if and only if A is fully connected (∞-connected) in $(E, \Delta(\mathbf{P}))$. A similar remark applies to a fully disconnected set.

The discrete analog of the previous result is given next.

6.1.12 Proposition. (*Multiscale Graph-Theoretic Connectivity*). Let $\mathcal{L} = \mathcal{P}(V)$ with the points as sup-generators, where V is a finite set. Let $G = (V, \sigma)$ be an $\overline{\mathbb{R}}$-fuzzy graph. Then, $\mathbf{C}$: $\mathbb{R} \to \mathcal{P}(\mathcal{P}(V))$, given by

$$\mathbf{C}(\tau) = \{U \subseteq V \mid U \text{ is } \tau\text{-connected in } G = (V, \sigma)\}, \quad \tau \in \mathbb{R}, \tag{6.13}$$

defines a connectivity pyramid on $\mathcal{P}(V)$. □

Proof. From Proposition 3.4.7, we have that $\mathbf{C}(\tau) = \{U \subseteq V \mid U \text{ is connected in } G_\tau = (V, X_\tau(\sigma))\}$, for $\tau \in \mathbb{R}$. It follows immediately that $\mathbf{C}(\tau)$ is a connectivity class, for all $\tau \in \mathbb{R}$, which shows axiom (i) of a connectivity pyramid. From the fact that G_{τ_1} is a sub-graph of G_{τ_2}, for $\tau_1 \geq \tau_2$, and the observation that a connected set in a subgraph G_1 of G_2 is connected in G_2, we conclude that $\mathbf{C}(\cdot)$ is a decreasing mapping, which shows axiom (ii) of a connectivity pyramid. We now show axiom (iii). Given $\tau \in \mathbb{R}$, the inclusion $\mathbf{C}(\tau) \subseteq \bigcap_{s<\tau} \mathbf{C}(s)$ follows directly from the fact that $\mathbf{C}(\cdot)$ is decreasing. We show the converse inclusion by establishing the contrapositive. Suppose that $U \notin \mathbf{C}(\tau)$; i.e., there is a pair of points $v, w \in U$ such that $s(\Pi) < \tau$, for all paths $\Pi \subseteq U$ between v and w. Since V is finite, we have that U is finite and the set of all paths in U is finite. This means that there is an s such that $s(\Pi) < s < \tau$, for all paths $\Pi \subseteq U$ between v and w, which implies that $U \notin \mathbf{C}(s)$. But this implies that $A \notin \bigcap_{s<\tau} \mathbf{C}(s)$, as required. Q.E.D.

Therefore, in a multiscale graph-theoretic connectivity framework, the connected sets at scale τ correspond to the τ-connected sets in $G = (V, \sigma)$, which in turn correspond to the connected sets in the graph $G_\tau = (V, X_\tau(\sigma))$, for $\tau \in \mathbb{R}$ (so that the τ-level graph of G defines connectivity at scale τ). Furthermore, it is easy to verify that A is fully connected, i.e., $A \in \bigcap_{\tau \in \mathbb{R}} \mathbf{C}(\tau)$, if and only if A is fully connected (∞-connected) in $G = (V, \sigma)$. A similar remark applies to a fully disconnected element.

We conclude this section by studying the multiscale analogs of connectivity openings and reconstruction operators (see Sections 4.1.2 and 4.1.3, respectively).

Given a multiscale connectivity system $(\varphi, \mathbf{C})$ on $\mathcal{L}$, the *σ-connectivity openings* associated with $(\varphi, \mathbf{C})$ are given by:

$$\gamma_{\sigma,x}(A) = \bigvee\{C \in \mathbf{C}(\sigma) \mid x \leq C \leq A\}, \quad \sigma \in \mathbb{R},\ x \in \mathcal{S}, \tag{6.14}$$

for $A \in \mathcal{L}$. It is clear that $\mathrm{Inv}(\gamma_{\sigma,x}) = \mathbf{C}(\sigma) \cap \mathcal{M}^*(x) = \{C \in \mathcal{L} \mid C \in \mathbf{C}(\sigma),\ C \geq x\}$, for $\sigma \in \mathbb{R}$ and $x \in \mathcal{S}$, so that

$$\mathbf{C}(\sigma) = \bigcup_{x \in \mathcal{S}} \mathrm{Inv}(\gamma_{\sigma,x}) = \bigcup_{x \in \mathcal{S}} \{\gamma_{\sigma,x}(A) \mid A \in \mathcal{L}\}, \quad \sigma \in \mathbb{R}. \tag{6.15}$$

Given a $\sigma \in \mathbb{R}$, a *σ-connected component* or *σ-grain* of $A \in \mathcal{L}$ is a σ-connected element $C \in \mathcal{L}$ such that $C \leq A$ and there is no σ-connected element $C' \in \mathcal{L}$ with $C \leq C' \leq A$. If C is a σ-connected component of A, we write $C \lessdot_\sigma A$. It is clear that, if $x \leq A$, then $\gamma_{\sigma,x}(A)$ is the σ-connected component of $A \in \mathcal{L}$, marked by x, and that the family of σ-grains of A is given by $\mathcal{C}_\sigma(A) = \{\gamma_{\sigma,x}(A) \mid x \leq A\}$, for $\sigma \in \mathbb{R}$. The mapping $\mathbf{c}_A$: $\mathbb{R} \times \mathcal{S}(A) \to \mathcal{L}$, given by $\mathbf{c}_A(\sigma, x) = \gamma_{\sigma,x}(A)$, for $\sigma \in \mathbb{R}$ and $x \in \mathcal{S}(A)$, is the *hierarchical partition of connected components* (HPCC) of A. Note that, for each $\sigma \in \mathbb{R}$, $\mathbf{c}_A(\sigma, \cdot)$ is a partition of A, in the sense of Definition 4.1.7. These are called the *σ-levels* or the *σ-partitions* of the HPCC $\mathbf{c}_A$ of A, for $\sigma \in \mathbb{R}$. We study these hierarchical partitions in more detail in Section 6.5.2.

Next, we give the multiscale version of Theorem 4.1.9. Recall the definition of the characteristic opening ψ° associated with an operator ψ, given by (2.15).

6.1.13 Theorem. Let $\mathcal{L}$ be a lattice with sup-generating family $\mathcal{S}$. For a given $\mathbf{C} \in \mathcal{Y}(\mathcal{L})$, let $\{\gamma_{\sigma,x} \mid \sigma \in \mathbb{R}, x \in \mathcal{S}\}$ be the σ-connectivity openings associated with $\mathbf{C}$, given by (6.14). Then,

(i) $\{\gamma_{\sigma,x} \mid x \in \mathcal{S}\}$ is a family of connectivity openings on $\mathcal{L}$, for each $\sigma \in \mathbb{R}$.

(ii) $\gamma_{\sigma,x} \leq \gamma_{\tau,x}$, if $\sigma \geq \tau$, for each $x \in \mathcal{S}$.

(iii) $\gamma_{\sigma,x} = \left(\bigwedge_{\tau<\sigma} \gamma_{\tau,x}\right)^\circ$, for each $\sigma \in \mathbb{R}$, $x \in \mathcal{S}$.

Conversely, let $\mathcal{X}(\mathcal{L})$ denote the set of all families of openings $\{\gamma_{\sigma,x} \mid \sigma \in \mathbb{R}, x \in \mathcal{S}\}$ that satisfy properties *(i)*–*(iii)* above. For $\{\gamma_{\sigma,x} \mid \sigma \in \mathbb{R}, x \in \mathcal{S}\} \in \mathcal{X}(\mathcal{L})$, let $\mathbf{C}$ be given by (6.15). Then, $\mathbf{C}$ is a connectivity pyramid on $\mathcal{L}$; i.e., $\mathbf{C} \in \mathcal{Y}(\mathcal{L})$. Moreover, its family of σ-connectivity openings coincides with $\{\gamma_{\sigma,x} \mid \sigma \in \mathbb{R}, x \in \mathcal{S}\}$. Hence, (6.14) and (6.15) establish a bijection between $\mathcal{Y}(\mathcal{L})$ and $\mathcal{X}(\mathcal{L})$. □

Proof. Property (i) is obvious. To show property (ii), note that, if $\sigma \geq \tau$, we have that $\mathrm{Inv}(\gamma_{\sigma,x}) = \mathbf{C}(\sigma) \cap \mathcal{M}^*(x) \subseteq \mathbf{C}(\tau) \cap \mathcal{M}^*(x) = \mathrm{Inv}(\gamma_{\tau,x})$, for each $x \in \mathcal{S}$. The desired result then follows from Proposition 2.2.1(a). To show property (iii), for given $\sigma \in \mathbb{R}$ and $x \in \mathcal{S}$, let $\psi = \bigwedge_{\tau<\sigma} \gamma_{\tau,x}$. Note that ψ is an increasing and anti-extensive operator. Moreover, $\mathrm{Inv}(\psi) = \bigcap_{\tau<\sigma} \mathrm{Inv}(\gamma_{\tau,x})$, due to the anti-extensivity of each $\gamma_{\tau,x}$. It follows from Corollary 2.2.7 that $\mathrm{Inv}(\psi^\circ) = \mathrm{Inv}(\psi) = \bigcap_{\tau<\sigma} \mathrm{Inv}(\gamma_{\tau,x}) = \bigcap_{\tau<\sigma}(\mathbf{C}(\tau) \cap \mathcal{M}^*(x)) = (\bigcap_{\tau<\sigma} \mathbf{C}(\tau)) \cap \mathcal{M}^*(x) = \mathbf{C}(\sigma) \cap \mathcal{M}^*(x) = \mathrm{Inv}(\gamma_{\sigma,x})$. But, from Proposition 2.2.5, ψ° is an opening, so that we can use Proposition 2.2.1(a) to conclude that $\gamma_{\sigma,x} = \psi^\circ = \left(\bigwedge_{\tau<\sigma} \gamma_{\tau,x}\right)^\circ$, as required.

Now, assume that $\{\gamma_{\sigma,x} \mid \sigma \in \mathbb{R}, x \in \mathcal{S}\} \in \mathcal{X}(\mathcal{L})$. We show that $\mathbf{C}$, given by (6.15), satisfies axioms (i)–(iii) of a connectivity pyramid. Axiom (i) follows directly from property (i). Axiom (ii) follows from property (ii): if $\sigma \geq \tau$, for each $x \in \mathcal{S}$, we have that $\gamma_{\sigma,x} \leq \gamma_{\tau,x} \Rightarrow \mathrm{Inv}(\gamma_{\sigma,x}) \subseteq \mathrm{Inv}(\gamma_{\tau,x}) \Rightarrow \mathbf{C}(\sigma) = \bigcup_{x\in\mathcal{S}} \mathrm{Inv}(\gamma_{\sigma,x}) \subseteq \bigcup_{x\in\mathcal{S}} \mathrm{Inv}(\gamma_{\tau,x}) = \mathbf{C}(\tau)$, where we used Proposition 2.2.1(a). To show axiom (iii), first note that, from property (i), we know that $\mathrm{Inv}(\gamma_{\sigma,x}) = \mathbf{C}(\sigma) \cap \mathcal{M}^*(x)$, for all $\sigma \in \mathbb{R}$. For given $\sigma \in \mathbb{R}$ and $x \in \mathcal{S}$, let $\psi = \bigwedge_{\tau<\sigma} \gamma_{\tau,x}$. From property (iii), we have that $\gamma_{\sigma,x} = \psi^\circ$, so that $\mathrm{Inv}(\gamma_{\sigma,x}) = \mathrm{Inv}(\psi^\circ) = \bigcap_{\tau<\sigma} \mathrm{Inv}(\gamma_{\tau,x}) = \bigcap_{\tau<\sigma}(\mathbf{C}(\tau) \cap \mathcal{M}^*(x)) = (\bigcap_{\tau<\sigma} \mathbf{C}(\tau)) \cap \mathcal{M}^*(x)$. Hence, we have that $\mathbf{C}(\sigma) = \bigcup_{x\in\mathcal{S}} \mathrm{Inv}(\gamma_{\sigma,x}) = \bigcup_{x\in\mathcal{S}}((\bigcap_{\tau<\sigma} \mathbf{C}(\tau)) \cap \mathcal{M}^*(x)) = (\bigcap_{\tau<\sigma} \mathbf{C}(\tau)) \cap \bigcup_{x\in\mathcal{S}} \mathcal{M}^*(x) = (\bigcap_{\tau<\sigma} \mathbf{C}(\tau)) \cap \mathcal{L} = \bigcap_{\tau<\sigma} \mathbf{C}(\tau)$, as required, where we used the fact that $\mathcal{P}(\mathcal{L})$ is an infinite $\vee$-distributive lattice. Finally, it follows from property (i) and Theorem 4.1.9 that the family of σ-connectivity openings associated with $\mathbf{C}$ coincides with $\{\gamma_{\sigma,x} \mid \sigma \in \mathbb{R}, x \in \mathcal{S}\}$. Q.E.D.

The previous theorem shows that a multiscale connectivity system on a lattice $\mathcal{L}$ can be equivalently specified by a family $\{\gamma_{\sigma,x} \mid \sigma \in \mathbb{R}, x \in \mathcal{S}\} \in \mathcal{X}(\mathcal{L})$. Therefore, a multiscale connectivity can be specified in three distinct equivalent ways, by means of a connectivity measure, a connectivity pyramid, or a family of σ-connectivity openings. Clearly, an element $A \in \mathcal{L}$ is σ-connected, for $\sigma \in \mathbb{R}$, if and only if $\gamma_{\sigma,x}(A) = A$, for all $x \leq A$. Similar remarks apply regarding full connectivity and full disconnectedness.

Property (i) of Theorem 6.1.13 states the fact that $\gamma_{\sigma,x}$ is the connectivity opening at scale σ. Property (ii) means that, for each $x \in \mathcal{S}$, the family of openings $\{\gamma_{\sigma,x} \mid \sigma \in \mathbb{R}\}$ constitutes a granulometry on $\mathcal{L}$, parameterized by scale. Furthermore, property (iii) says that this granulometry satisfies a smoothness constraint, namely, that the connectivity opening $\gamma_{\sigma,x}$ is the greatest opening that is smaller than $\bigwedge_{\tau<\sigma} \gamma_{\tau,x}$, for each $\sigma \in \mathbb{R}$ (see Corollary 2.2.7).

Note that, in general, it is not true that $\gamma_{\sigma,x} = \bigwedge_{\tau<\sigma} \gamma_{\tau,x}$, for $\sigma \in \mathbb{R}$ and $x \in \mathcal{S}$. Clearly, that would happen if and only if $\bigwedge_{\tau<\sigma} \gamma_{\tau,x}$ were an opening on $\mathcal{L}$, for $\sigma \in \mathbb{R}$ and $x \in \mathcal{S}$. But it is a well-known fact that the infimum of openings is not necessarily an opening. For a counterexample, let $\mathcal{L} = \mathcal{P}(\overline{\mathbb{R}})$ with the points as sup-generators. Let $\mathbf{C}\colon \mathbb{R} \to \mathcal{P}(\mathcal{L})$ be given by

$$\mathbf{C}(\sigma) = \emptyset \cup \mathcal{S} \cup \{\, (\tau, \infty] \mid \tau \geq \sigma\}, \quad \sigma \in \mathbb{R}. \tag{6.16}$$

It is easy to check that $\mathbf{C}$ is a connectivity pyramid on $\mathcal{P}(\overline{\mathbb{R}})$. Note that, in this case,

$$\gamma_{\sigma,x}(\overline{\mathbb{R}}) = \begin{cases} (\sigma, \infty], & \text{if } \sigma < \text{val}(x) \\ x, & \text{otherwise} \end{cases}, \quad \sigma \in \mathbb{R},\ x \in \mathcal{S}, \tag{6.17}$$

where val(x) denotes the numerical value associated with point x. Pick a point x such that val$(x) \in \mathbb{R}$, and let $\sigma <$ val(x). From (6.17), we have that $\gamma_{\sigma,x}(\overline{\mathbb{R}}) = (\sigma, \infty] \neq [\sigma, \infty] = \bigcap_{\tau<\sigma} (\tau, \infty] = \bigcap_{\tau<\sigma} \gamma_{\tau,x}(\overline{\mathbb{R}})$, so that $\gamma_{\sigma,x} \neq \bigwedge_{\tau<\sigma} \gamma_{\tau,x}$ (in particular, $\bigwedge_{\tau<\sigma} \gamma_{\tau,x}$ is not an opening on $\mathcal{L}$).

The following result provides a case where the equality $\gamma_{\sigma,x} = \bigwedge_{\tau<\sigma} \gamma_{\tau,x}$, for $a \in \mathbb{R}$ and $x \in \mathcal{S}$, *does* hold. Recall the concept of $\downarrow$-continuous operators, defined in Section 2.2.

6.1.14 Proposition. Let $\mathcal{L}$ be a lattice, and let $\{\gamma_{\sigma,x} \mid \sigma \in \mathbb{R}, x \in \mathcal{S}\} \in \mathcal{X}(\mathcal{L})$ be a family of $\downarrow$-continuous σ-connectivity openings on $\mathcal{L}$. We have that $\gamma_{\sigma,x} = \bigwedge_{\tau<\sigma} \gamma_{\tau,x}$, for $\sigma \in \mathbb{R}$ and $x \in \mathcal{S}$. □

Proof. Given $\sigma \in \mathbb{R}$ and $x \in \mathcal{S}$, let $\psi = \bigwedge_{\tau<\sigma} \gamma_{\tau,x}$. As argued before, it suffices to show that ψ is an opening on $\mathcal{L}$. It is clear that ψ is increasing and anti-extensive. We show that ψ is idempotent. Given $A \in \mathcal{L}$, we have that $\psi\psi(A) = \bigwedge_{\tau<\sigma} \gamma_{\tau,x}(\bigwedge_{\tau'<\sigma} \gamma_{\tau',x}(A)) = \bigwedge_{\tau<\sigma} \gamma_{\tau,x}(\bigwedge_{\tau\leq\tau'<\sigma} \gamma_{\tau',x}(A))$, since $\{\gamma_{\tau',x}(A) \mid \tau' < \sigma\}$ is a decreasing family. Since each opening $\gamma_{\tau,x}$ is $\downarrow$-continuous, it follows from Proposition 2.2.10 that $\psi\psi(A) = \bigwedge_{\tau<\sigma} \bigwedge_{\tau\leq\tau'<\sigma} \gamma_{\tau,x}\gamma_{\tau',x}(A) = \bigwedge_{\tau<\sigma} \bigwedge_{\tau\leq\tau'<\sigma} \gamma_{\tau',x}(A) = \bigwedge_{\tau'<\sigma} \gamma_{\tau',x}(A) = \psi(A)$, where we used the fact that $\gamma_{\tau,x}\gamma_{\tau',x}(A) = \gamma_{\tau',x}(A)$, for all $\tau \leq \tau'$, which follows from the fact that $\{\gamma_{\sigma,x} \mid \sigma \in \mathbb{R}\}$ is a granulometry on $\mathcal{L}$ and Proposition 2.2.9. Q.E.D.

We remark that, if E is a finite set, the σ-connectivity openings associated with a multiscale connectivity on $\mathcal{L} = \mathcal{P}(E)$ are trivially $\downarrow$-continuous, and therefore always satisfy the smoothness property $\gamma_{\sigma,x} = \bigwedge_{\tau<\sigma} \gamma_{\tau,x}$, for $\sigma \in \mathbb{R}$ and $x \in \mathcal{S}$. For example, the connectivity openings associated with multiscale graph-theoretic connectivities (see Proposition 6.1.12) satisfy this property. In Section 6.3, we study examples of multiscale connectivity with non-trivial $\downarrow$-continuous σ-connectivity openings.

Given a marker $M \in \mathcal{L}$, the *σ-reconstruction* $\rho_\sigma(A \mid M)$ of $A \in \mathcal{L}$ from M is defined by:

$$\rho_\sigma(A \mid M) = \bigvee_{x \leq M} \gamma_{\sigma,x}(A), \quad \sigma \in \mathbb{R}. \tag{6.18}$$

As in the single-scale case, it is easy to see that

$$\gamma_{\sigma,x}(A) = \begin{cases} \rho_\sigma(A \mid x), & \text{if } \; x \leq A \\ O, & \text{otherwise} \end{cases}, \quad \sigma \in \mathbb{R},\, x \in \mathcal{S}, \tag{6.19}$$

for $A \in \mathcal{L}$. It follows easily from the corresponding single-scale result that $\rho_\sigma(A \mid M) = \bigvee\{C \lessdot_\sigma A \mid C \wedge M \neq O\}$; i.e., $\rho_\sigma(A \mid M)$ extracts the σ-connected components of A that "intersect" marker M.

Being a supremum of openings, the operator $\rho_\sigma(\cdot \mid M)$ is an opening on $\mathcal{L}$, for $\sigma \in \mathbb{R}$ and a fixed marker $M \in \mathcal{L}$. Clearly, $\rho_\sigma(A \mid M)$ is the reconstruction of A from M at scale σ. In addition, it is clear that $\rho_\sigma(\cdot \mid M) \leq \rho_\tau(\cdot \mid M)$, for $\sigma \geq \tau$; i.e., for each fixed marker $M \in \mathcal{L}$, the family of openings $\{\rho_\sigma(\cdot \mid M) \mid \sigma \in \mathbb{R}\}$ constitutes a granulometry on $\mathcal{L}$, parameterized by scale. However, this granulometry does not satisfy a smoothness constraint such as the one in property (*iii*) of Theorem 6.1.13; i.e., it is not true in general that $\rho_\sigma(\cdot \mid M) = \left(\bigwedge_{\tau<\sigma} \rho_\tau(\cdot \mid M)\right)^\circ$, for each $\sigma \in \mathbb{R}$ and $M \in \mathcal{L}$. For a counterexample, let $\mathcal{L} = \mathcal{P}(\overline{\mathbb{R}})$, with the points as sup-generators, and let $\mathbf{C}$ be the connectivity pyramid on $\mathcal{P}(\overline{\mathbb{R}})$ defined in (6.16). For a given $\sigma \in \mathbb{R}$, let $M = [-\infty, \sigma]$. It is easy to check that $\rho_\tau(\overline{\mathbb{R}} \mid M) = \overline{\mathbb{R}}$, for all $\tau < \sigma$; i.e., $\overline{\mathbb{R}} \in \bigcap_{\tau<\sigma} \operatorname{Inv}(\rho_\tau(\cdot \mid M))$. However, $\rho_\sigma(\overline{\mathbb{R}} \mid M) = M \neq \overline{\mathbb{R}}$, so that $\overline{\mathbb{R}} \notin \operatorname{Inv}(\rho_\sigma(\cdot \mid M))$. In other words, $\operatorname{Inv}(\rho_\sigma(\cdot \mid M)) \neq \bigcap_{\tau<\sigma} \operatorname{Inv}(\rho_\tau(\cdot \mid M))$. It easily follows that $\rho_\sigma(\cdot \mid M) \neq \left(\bigwedge_{\tau<\sigma} \rho_\tau(\cdot \mid M)\right)^\circ$.

6.2 Discrete Multiscale Connectivity

In the case of digital images, the set of scales available for multiscale analysis is usually discrete. In this section, we show that it is possible to specialize all multiscale connectivity notions discussed in the previous section to the discrete case. Furthermore, the analysis becomes simpler since, in this case, the semi-continuity axiom (*iii*) of Definition 6.1.7 associated with the levels of a connectivity pyramid is obsolete. We remark here that only the set of scales is required to be discrete, whereas the underlying lattice may be arbitrary. In practice, however, both the set of scales and the underlying lattice are discrete.

Rather than simply converting all the concepts and results discussed in the last section to the discrete setting, we focus our attention here on discretizing the main ideas related to multiscale connectivity. We begin with the definition of a discrete connectivity measure.

6.2.1 Definition. Let $\mathcal{L}$ be a lattice and φ be a connectivity measure on $\mathcal{L}$. If φ takes only integer values (i.e., if $\varphi(\mathcal{L}) \subseteq \overline{\mathbb{Z}}$), then φ is said to be a *discrete connectivity measure* on $\mathcal{L}$. △

Discrete connectivity measures are of course special cases of connectivity measures. In particular, notions defined in the last section regarding connectivity measures, such as full connectivity, full disconnectedness, σ-connectivity, σ-sections, and strong and translation-invariant connectivity measures, apply to the discrete case as well. As a matter of fact, some of the examples of connectivity measure discussed in the last section are in fact discrete; e.g., the simple binary connectivity measure associated with single-scale connectivity classes and the strong translation-invariant connectivity measure of Example 6.1.2.

The following is the discrete analog of Proposition 6.1.3.

6.2.2 Proposition. If φ is a discrete connectivity measure on a lattice $\mathcal{L}$, and $f\colon \overline{\mathbb{Z}} \to \overline{\mathbb{Z}}$ is a nondecreasing function such that $f(\infty) = \infty$, then the composition $f(\varphi(\cdot))$ is a discrete connectivity measure on $\mathcal{L}$. □

The previous result has the same useful consequences as its continuous counterpart. For example, if φ is a given discrete connectivity measure on $\mathcal{L}$ and $a, b \in \mathbb{Z}$, with $a > 0$, then $a\varphi + b$ is also a connectivity measure on $\mathcal{L}$.

Next, we give the discrete version of Proposition 6.1.5.

6.2.3 Proposition. Let $\{\varphi_\alpha\}$ be an arbitrary family of discrete connectivity measures on a lattice $\mathcal{L}$. The pointwise infimum $(\bigwedge \varphi_\alpha)(A) = \bigwedge \varphi_\alpha(A)$, for $A \in \mathcal{L}$, is a discrete connectivity measure on $\mathcal{L}$. □

We remark that, as in the continuous case, the pointwise supremum of discrete connectivity measures is not in general a discrete connectivity measure (as a matter of fact, the connectivity measures used in the counterexample given after Proposition 6.1.5 are actually discrete connectivity measures).

For a fixed sup-generating family, let $\mathcal{M}_d(\mathcal{L})$ be the set of all discrete connectivity measures defined on a lattice $\mathcal{L}$. The following is the discrete analog of Proposition 6.1.6.

6.2.4 Proposition. Given a lattice $\mathcal{L}$, $\mathcal{M}_d(\mathcal{L})$ is a lattice, under the product partial order.□

We have the following further characterization of $\mathcal{M}_d(\mathcal{L})$.

6.2.5 Proposition. Given a lattice $\mathcal{L}$, $\mathcal{M}_d(\mathcal{L})$ is a sublattice of $\mathcal{M}(\mathcal{L})$, under the product partial order. □

PROOF. Clearly, $\mathcal{M}_d(\mathcal{L})$ is a non-empty subset of $\mathcal{M}(\mathcal{L})$. In addition, it follows from Proposition 6.2.3 that, under the product partial order, the infimum in $\mathcal{M}_d(\mathcal{L})$ is given by the pointwise infimum in $\mathcal{M}(\mathcal{L})$. It remains to show that, under the product partial order, the supremum in $\mathcal{M}_d(\mathcal{L})$ coincides with the supremum in $\mathcal{M}(\mathcal{L})$. Let $\{\varphi_\alpha\}$ be a family of discrete connectivity measures in $\mathcal{M}_d(\mathcal{L})$, and $\overline{\varphi} = \bigwedge\{\varphi \in \mathcal{M}(\mathcal{L}) \mid \varphi \geq \bigvee \varphi_\alpha\}$ be the supremum of $\{\varphi_\alpha\}$ in $\mathcal{M}(\mathcal{L})$. We need to show that $\overline{\varphi} \in \mathcal{M}_d(\mathcal{L})$. Suppose that it is not; then, there is an $A_0 \in \mathcal{L}$ such that $\overline{\varphi}(A_0) \notin \overline{\mathbb{Z}}$. Define a connectivity measure $\varphi' \in \mathcal{M}(\mathcal{L})$ by setting $\varphi'(A) = \overline{\varphi}(A)$, for $A \neq A_0$, and $\varphi'(A_0) = \lfloor \overline{\varphi}(A_0) \rfloor$, where $\lfloor a \rfloor$ is the greatest integer less than or equal to a. It is clear that $\varphi' \geq \bigvee \varphi_\alpha$, with $\varphi' < \overline{\varphi}$, a contradiction, since $\overline{\varphi}$ was assumed to be the supremum of $\{\varphi_\alpha\}$ in $\mathcal{M}(\mathcal{L})$. Therefore, $\overline{\varphi} \in \mathcal{M}_d(\mathcal{L})$, as required. Q.E.D.

We now define the discrete analog of a connectivity pyramid.

6.2.6 Definition. Let $\mathcal{L}$ be a lattice with sup-generating family $\mathcal{S}$. A *discrete connectivity pyramid* on $\mathcal{L}$ is a mapping $\mathbf{C}$: $\mathbb{Z} \to \mathcal{P}(\mathcal{L})$ such that:

(*i*) $\mathbf{C}(\sigma)$ is a connectivity class in $\mathcal{L}$, for each $\sigma \in \mathbb{Z}$,

(*ii*) $\mathbf{C}(\sigma) \subseteq \mathbf{C}(\tau)$, if $\sigma \geq \tau$. △

As in the continuous case, the σ-connectivity class $\mathbf{C}(\sigma)$ corresponds to connectivity at scale σ. But the set of scales is now discrete, which makes obsolete axiom (*iii*) of Definition 6.1.7. Notions defined in the last section regarding connectivity pyramids, such as full connectivity, full disconnectedness, σ-connectivity, σ-levels, and strong and translation-invariant connectivity pyramids, apply to the discrete case as well, with the obvious modifications.

The set of all discrete connectivity pyramids on a lattice $\mathcal{L}$ is denoted by $\mathcal{Y}_d(\mathcal{L})$. We mention that a similar result to Proposition 6.1.8 can be obtained regarding $\mathcal{Y}_d(\mathcal{L})$. However, we limit ourselves here to the following proposition.

6.2.7 Proposition. Given a lattice $\mathcal{L}$, the set $\mathcal{Y}_d(\mathcal{L})$ is a lattice, under the product inclusion order, with infimum $(\bigwedge \mathbf{C}_\alpha)(\sigma) = \bigcap \mathbf{C}_\alpha(\sigma)$, for $\sigma \in \mathbb{Z}$, and supremum $(\bigvee \mathbf{C}_\alpha)(\sigma) = \phi(\bigcup \mathbf{C}_\alpha(\sigma))$, for $\sigma \in \mathbb{Z}$, where ϕ is the closing on $\mathcal{P}(\mathcal{L})$ given by (4.7). □

The following proposition provides a relationship between $\mathcal{Y}_d(\mathcal{L})$ and the lattice $\mathcal{Y}(\mathcal{L})$ of continuous connectivity pyramids.

6.2.8 Proposition. Given a lattice $\mathcal{L}$, the lattice $\mathcal{Y}_d(\mathcal{L})$ is isomorphic to a sublattice of $\mathcal{Y}(\mathcal{L})$, under the product inclusion order. Moreover, the isomorphism $\Lambda : \mathcal{Y}_d(\mathcal{L}) \to \mathcal{Y}(\mathcal{L})$ is given by

$$\Lambda(\mathbf{C})(\sigma) = \mathbf{C}(\lceil \sigma \rceil), \quad \sigma \in \mathbb{R}, \tag{6.20}$$

where $\lceil a \rceil$ is the smallest integer greater than or equal to a. □

Proof. Note that $\Lambda(\mathbf{C})$, for $\mathbf{C} \in \mathcal{Y}_d(\mathcal{L})$, is a connectivity pyramid on $\mathcal{L}$, since axioms (i) and (ii) of Definition 6.1.7 are clearly satisfied, while axiom (iii) follows easily from the right-continuity of the "ceiling" function $f(a) = \lceil a \rceil$, for $a \in \overline{\mathbb{R}}$. Therefore, Λ defines a mapping from $\mathcal{Y}_d(\mathcal{L})$ into $\mathcal{Y}(\mathcal{L})$. This mapping is clearly injective, and it therefore establishes a bijection between $\mathcal{Y}_d(\mathcal{L})$ and the range $\Lambda(\mathcal{Y}_d(\mathcal{L}))$, with inverse Λ^{-1} from $\Lambda(\mathcal{Y}_d(\mathcal{L}))$ into $\mathcal{Y}_d(\mathcal{L})$ defined accordingly. It is clear that Λ is order preserving; i.e., $\mathbf{C} \leq \mathbf{C}' \Leftrightarrow \Lambda(\mathbf{C}) \leq \Lambda(\mathbf{C}')$, for $\mathbf{C}, \mathbf{C}' \in \mathcal{Y}_d(\mathcal{L})$, where the partial order in $\Lambda(\mathcal{Y}_d(\mathcal{L}))$ is the product inclusion order. Therefore, $\Lambda(\mathcal{Y}_d(\mathcal{L}))$ is a complete lattice under the product inclusion order, which is isomorphic to $\mathcal{Y}_d(\mathcal{L})$ via Λ. It remains to show that $\Lambda(\mathcal{Y}_d(\mathcal{L}))$ is a sublattice of $\mathcal{Y}(\mathcal{L})$. Clearly, $\mathcal{Y}_d(\mathcal{L})$ is a partially ordered subset of $\mathcal{Y}(\mathcal{L})$, under the product inclusion order. Let $\{\mathbf{C}_\alpha\}$ be a family of connectivity pyramids in $\Lambda(\mathcal{Y}_d(\mathcal{L}))$, and $\{\mathbf{C}'_\alpha = \Lambda^{-1}(\mathbf{C}_\alpha)\}$ be the corresponding family in $\mathcal{Y}_d(\mathcal{L})$. The infimum of $\{\mathbf{C}_\alpha\}$ in $\Lambda(\mathcal{Y}_d(\mathcal{L}))$ is given by $\Lambda(\bigwedge \mathbf{C}'_\alpha)$, where $\bigwedge \mathbf{C}'_\alpha$ is given by a pointwise intersection in $\mathcal{Y}_d(\mathcal{L})$ (see Proposition 6.2.7). It can be easily checked that this coincides with the infimum of $\{\mathbf{C}_\alpha\}$ in $\mathcal{Y}(\mathcal{L})$. Similarly, the supremum of $\{\mathbf{C}_\alpha\}$ in $\Lambda(\mathcal{Y}_d(\mathcal{L}))$ is given by $\Lambda(\bigvee \mathbf{C}'_\alpha)$, where $(\bigvee \mathbf{C}'_\alpha)(\sigma) = \phi(\bigvee \mathbf{C}_\alpha(\sigma))$, for $\sigma \in \mathbb{Z}$ (see Proposition 6.2.7). It can be easily checked that this coincides with the supremum of $\{\mathbf{C}_\alpha\}$ in $\mathcal{Y}(\mathcal{L})$. Hence, $\Lambda(\mathcal{Y}_d(\mathcal{L}))$ is a sublattice of $\mathcal{Y}_d(\mathcal{L})$, as required. Q.E.D.

Being isomorphic to a sublattice of $\mathcal{Y}(\mathcal{L})$, lattice $\mathcal{Y}_d(\mathcal{L})$ is *embedded* in $\mathcal{Y}(\mathcal{L})$, via Λ. Note also that $\Lambda(\mathcal{Y}_d(\mathcal{L}))$ comprises those connectivity pyramids that are discrete in nature; i.e., their σ-level connectivity classes change only at the integers. In this sense, discrete connectivity pyramids can be thought of as special cases of connectivity pyramids.

As in the continuous case, discrete connectivity pyramids are closely related to discrete connectivity measures. This is shown by the next result, which is the discrete analog of Theorem 6.1.9.

6.2.9 Theorem. Let $\mathcal{L}$ be a lattice with sup-generating family $\mathcal{S}$. The lattice $\mathcal{M}_d(\mathcal{L})$ of discrete connectivity measures on $\mathcal{L}$ is isomorphic to the lattice $\mathcal{Y}_d(\mathcal{L})$ of discrete connectivity pyramids on $\mathcal{L}$. Moreover, the isomorphism $\Gamma_d : \mathcal{M}_d(\mathcal{L}) \to \mathcal{Y}_d(\mathcal{L})$ is given by

$$\Gamma_d(\varphi)(\sigma) = \{A \in \mathcal{L} \mid \varphi(A) \geq \sigma\}, \quad \sigma \in \mathbb{Z}, \tag{6.21}$$

with inverse $\Gamma_d^{-1} : \mathcal{Y}_d(\mathcal{L}) \to \mathcal{M}_d(\mathcal{L})$, given by

$$\Gamma_d^{-1}(\mathbf{C})(A) = \bigvee\{\sigma \in \mathbb{Z} \mid A \in \mathbf{C}(\sigma)\}, \quad A \in \mathcal{L}. \tag{6.22}$$

□

The isomorphism between lattices $\mathcal{M}_d(\mathcal{L})$ and $\mathcal{Y}_d(\mathcal{L})$ is of course a bijection; i.e., to each discrete connectivity measure φ on $\mathcal{L}$, there is an associated equivalent discrete connectivity pyramid $\mathbf{C}$ on $\mathcal{L}$, which consists of the σ-sections of φ. Conversely, φ can be regenerated by "stacking up" the σ-levels of $\mathbf{C}$. Therefore, it is convenient to say that $\mathcal{L}$ is furnished with a *discrete multiscale connectivity system* $(\varphi, \mathbf{C}) \in \mathcal{M}_d(\mathcal{L}) \times \mathcal{Y}_d(\mathcal{L})$, such that φ and $\mathbf{C}$ are equivalent under the bijection given in Theorem 6.2.9. As in the continuous case, we have that $\varphi(A) \geq \sigma \;\Leftrightarrow\; A \in \mathbf{C}(\sigma)$, in which case A is σ-connected, for $\sigma \in \mathbb{Z}$. In addition, $\varphi(A) = \infty \;\Leftrightarrow\; A \in \bigcap_{\sigma \in \mathbb{Z}} \mathbf{C}(\sigma)$, in which case A is fully connected. Similarly, $\varphi(A) = -\infty \;\Leftrightarrow\; A \notin \bigcup_{\sigma \in \mathbb{Z}} \mathbf{C}(\sigma)$, in which case A is fully disconnected. Similarly to the continuous case, one can also define strong and translation-invariant discrete multiscale connectivity systems.

Next, we give two important examples of discrete multiscale connectivity.

6.2.10 Example.

(a) Let $\mathcal{L} = \mathcal{P}(E)$ with the points as sup-generators, and $\mathbf{P} = \{\mathcal{G}_\tau \mid \tau \in \mathbb{Z}\}$ be a decreasing family of topologies on E. The family $\mathbf{P}$ is a *discrete topology pyramid*, with τ-level topologies $\mathcal{G}_\tau$, for $\tau \in \mathbb{Z}$. The mapping $\mathbf{C}$: $\mathbb{Z} \to \mathcal{P}(\mathcal{P}(E))$, given by

$$\mathbf{C}(\tau) = \{A \subseteq E \mid A \text{ is connected in } (E, \mathcal{G}_{-\tau})\}, \quad \tau \in \mathbb{Z}, \tag{6.23}$$

is a discrete connectivity pyramid on $\mathcal{P}(E)$. This is the discrete analog of multiscale topological connectivity (see Proposition 6.1.11).

(b) Let $\mathcal{L} = \mathcal{P}(V)$ with the points as sup-generators, where V is a finite set. Let $\{G_\tau = (V, L_\tau) \mid \tau \in \mathbb{Z}\}$ be a family of graphs defined on V, where $L_{\tau_1} \subseteq L_{\tau_2}$, for $\tau_1 \geq \tau_2$. The mapping $\mathbf{C}$: $\mathbb{Z} \to \mathcal{P}(\mathcal{P}(E))$, given by

$$\mathbf{C}(\tau) = \{U \subseteq V \mid U \text{ is connected in } G_\tau = (V, L_\tau)\}, \quad \tau \in \mathbb{Z}, \tag{6.24}$$

is a discrete connectivity pyramid on $\mathcal{P}(E)$. This is the discrete analog of multiscale graph-theoretic connectivity (see Proposition 6.1.12). $\Diamond$

We conclude this section with a discussion of the discrete analogs of σ-connectivity openings and σ-reconstruction operators. The analysis is substantially simplified in this case, due to the discrete nature of the scale parameter.

Given a discrete multiscale connectivity system $(\varphi, \mathbf{C})$ on $\mathcal{L}$, the (discrete) σ-connectivity openings associated with $(\varphi, \mathbf{C})$ are defined exactly as in the continuous case; i.e.,

$$\gamma_{\sigma,x}(A) = \bigvee\{C \in \mathbf{C}(\sigma) \mid x \leq C \leq A\}, \quad \sigma \in \mathbb{Z},\ x \in \mathcal{S}, \tag{6.25}$$

for $A \in \mathcal{L}$. We also have that

$$\mathbf{C}(\sigma) = \bigcup_{x \in \mathcal{S}} \mathrm{Inv}(\gamma_{\sigma,x}) = \bigcup_{x \in \mathcal{S}} \{\gamma_{\sigma,x}(A) \mid A \in \mathcal{L}\}, \quad \sigma \in \mathbb{Z}. \tag{6.26}$$

The following result is the discrete analog of Theorem 6.1.13.

6.2.11 Theorem. Let $\mathcal{L}$ be a lattice with sup-generating family $\mathcal{S}$. For a given $\mathbf{C} \in \mathcal{Y}_d(\mathcal{L})$, let $\{\gamma_{\sigma,x} \mid \sigma \in \mathbb{Z}, x \in \mathcal{S}\}$ be the σ-connectivity openings associated with $\mathbf{C}$, given by (6.25). Then,

(i) $\{\gamma_{\sigma,x} \mid x \in \mathcal{S}\}$ is a family of connectivity openings on $\mathcal{L}$, for each $\sigma \in \mathbb{Z}$.

(ii) $\gamma_{\sigma,x} \leq \gamma_{\tau,x}$, if $\sigma \geq \tau$, for each $x \in \mathcal{S}$.

Conversely, let $\mathcal{X}_d(\mathcal{L})$ denote the set of all families of openings $\{\gamma_{\sigma,x} \mid \sigma \in \mathbb{Z}, x \in \mathcal{S}\}$ that satisfy properties (i) and (ii) above. For $\{\gamma_{\sigma,x} \mid \sigma \in \mathbb{Z}, x \in \mathcal{S}\} \in \mathcal{X}_d(\mathcal{L})$, let $\mathbf{C}$ be given by (6.26). Then, $\mathbf{C}$ is a discrete connectivity pyramid on $\mathcal{L}$; i.e., $\mathbf{C} \in \mathcal{Y}_d(\mathcal{L})$. Moreover, its family of σ-connectivity openings coincides with $\{\gamma_{\sigma,x} \mid \sigma \in \mathbb{Z}, x \in \mathcal{S}\}$. Hence, (6.25) and (6.26) establish a bijection between $\mathcal{Y}_d(\mathcal{L})$ and $\mathcal{X}_d(\mathcal{L})$. □

The previous theorem says that a discrete multiscale connectivity on a lattice $\mathcal{L}$ corresponds, in a unique fashion, to a family $\{\gamma_{\sigma,x} \mid \sigma \in \mathbb{Z}, x \in \mathcal{S}\} \in \mathcal{X}_d(\mathcal{L})$. Properties (i) and (ii) mean that $\gamma_{\sigma,x}$ is the connectivity opening at scale σ, for each $x \in \mathcal{S}$, and $\{\gamma_{\sigma,x} \mid \sigma \in \mathbb{Z}\}$ constitutes a granulometry on $\mathcal{L}$ parameterized by the scale parameter σ, respectively.

The concept of a (discrete) σ-connected component is defined exactly as in the continuous case. If C is a σ-connected component of A, we write $C \lessdot_\sigma A$, as before. All remarks made in the last section regarding σ-connected components apply to the discrete case as well.

Given a marker $M \in \mathcal{L}$, the (discrete) σ-reconstruction $\rho_\sigma(A \mid M)$ of $A \in \mathcal{L}$ from M is defined by:

$$\rho_\sigma(A \mid M) = \bigvee_{x \leq M} \gamma_{\sigma,x}(A), \quad \sigma \in \mathbb{Z}. \tag{6.27}$$

As in the continuous case, it is easy to see that

$$\gamma_{\sigma,x}(A) = \begin{cases} \rho_\sigma(A \mid x), & \text{if } x \leq A \\ O, & \text{otherwise} \end{cases}, \quad \sigma \in \mathbb{Z},\ x \in \mathcal{S}, \tag{6.28}$$

for $A \in \mathcal{L}$.

The discrete nature of the scale parameter allows one to establish the following result, which can be thought of as the discrete multiscale version of Theorem 4.1.17.

6.2.12 Theorem. Let $\mathcal{L}$ be an infinite $\vee$-distributive lattice with sup-generating family $\mathcal{S}$. For a family of σ-connectivity openings $\{\gamma_{\sigma,x} \mid \sigma \in \mathbb{Z}, x \in \mathcal{S}\} \in \mathcal{X}_d(\mathcal{L})$, let $\{\rho_\sigma \mid \sigma \in \mathbb{Z}\}$ be its family of σ-reconstruction operators, given by (6.27). Then,

(i) $\rho_\sigma : \mathcal{L} \times \mathcal{L} \to \mathcal{L}$ is a reconstruction operator that satisfies properties (i)–(v) of Theorem 4.1.17, for each $\sigma \in \mathbb{Z}$.

(ii) $\rho_\sigma(\cdot \mid M) \leq \rho_\tau(\cdot \mid M)$, if $\sigma \geq \tau$, for each $M \in \mathcal{L}$.

Conversely, let $\mathcal{W}_d(\mathcal{L})$ denote the set of all families of operators $\{\rho_\sigma \mid \sigma \in \mathbb{Z}\}$ that satisfy properties (i) and (ii) above. For $\{\rho_\sigma \mid \sigma \in \mathbb{Z}\} \in \mathcal{W}_d(\mathcal{L})$, we have that $\{\gamma_{\sigma,x} \mid \sigma \in \mathbb{Z}, x \in \mathcal{S}\}$, defined by (6.28), belongs to $\mathcal{X}_d(\mathcal{L})$. Moreover, its family of σ-reconstruction operators coincides with $\{\rho_\sigma \mid \sigma \in \mathbb{Z}\}$. Hence, (6.27) and (6.28) establish a bijection between $\mathcal{X}_d(\mathcal{L})$ and $\mathcal{W}_d(\mathcal{L})$. □

Properties (i) and (ii) mean that ρ_σ is the reconstruction operator at scale σ and that, for each $M \in \mathcal{L}$, $\{\rho_\sigma(\cdot \mid M) \mid \sigma \in \mathbb{Z}\}$ constitutes a granulometry on $\mathcal{L}$ parameterized by the scale parameter σ. Together with Theorem 6.2.11, the previous result says that a discrete multiscale connectivity on an infinite $\vee$-distributive lattice $\mathcal{L}$ corresponds, in a unique fashion, to a family $\{\rho_\sigma \mid \sigma \in \mathbb{Z}\} \in \mathcal{W}_d(\mathcal{L})$.

A discrete multiscale connectivity on an infinite $\vee$-distributive lattice can therefore be specified in four distinct equivalent ways: by means of a discrete connectivity measure, a discrete connectivity pyramid, a family of discrete σ-connectivity openings, or a family of discrete σ-reconstruction operators.

6.3 Multiscale Connectivities Based on Multiscale Morphological Operators

In Mathematical Morphology, there are several examples of operations that have a natural multiscale interpretation, such as dilations by a scalable structuring element and granulometries [34, 76, 77]. In this section, we define and study examples of multiscale connectivities generated by such operations. We consider two general classes of multiscale operators, namely, clustering pyramids, which lead to "negative" multiscale connectivities, and contraction pyramids, which lead to "positive" multiscale connectivities, in a sense that will become clear in the sequel.

6.3.1 Multiscale Connectivities Generated by Clustering Pyramids

In this subsection, we formalize the notion of clustering pyramids, and we show how they can be used to construct multiscale connectivities. We will see that these multiscale connectivities are "negative" in nature.

We start with the following definition.

6.3.1 Definition. Let J be a poset. A family $\{\psi_\alpha \mid \alpha \in J\}$ of operators on a lattice $\mathcal{L}$ is said to have the *interpolation property* if:

$$\text{for } \alpha, \beta \in J, \text{ with } \alpha \geq \beta, \text{ there exists a } \gamma \in J \text{ such that } \psi_\alpha = \psi_\gamma \psi_\beta. \tag{6.29}$$

△

In other words, there is always an operator in the family that provides the necessary composition to go “up” from one operator in the family to another. In [75], families of operators with the interpolation property are referred to as pyramids of operators. However, we prefer to reserve this term here for a different, more restricted concept, which will be discussed below.

Next, we give classical examples of families of morphological operators that have the interpolation property. In what follows, $\mathbb{R}_+$ and $\mathbb{Z}_+$ denote the set of nonnegative real and nonnegative integer numbers, respectively.

6.3.2 Example.

(a) (*Dilation by a scalable structuring element*). Let $\mathcal{L} = \mathcal{P}(\mathbb{R}^n)$ and $J = \mathbb{R}_+$. We define the *scaled replicas* of a structuring element $B \subseteq \mathbb{R}^n$ by

$$\sigma B = \{v \in \mathbb{R}^n \mid v = \sigma w,\ w \in B\}, \tag{6.30}$$

for $\sigma \in \mathbb{R}_+$. Consider now the family $\{\delta_\sigma(A) = A \oplus \sigma B \mid \sigma \in \mathbb{R}_+\}$ of dilations, where B is a convex structuring element. From [34, Prop. 9.2], it follows that, if $\sigma \geq \tau$, then $\delta_\sigma(A) = A \oplus \sigma B = (A \oplus \tau B) \oplus (\sigma - \tau)B = \delta_{\sigma-\tau}\delta_\tau(A)$, so that (6.29) is satisfied, and $\{\delta_\sigma(A) = A \oplus \sigma B \mid \sigma \in \mathbb{R}_+\}$ has the interpolation property.

(b) (*Discrete dilation by a scalable structuring element*). This is the discrete analog of the previous example. Here, $\mathcal{L} = \mathcal{P}(\mathbb{Z}^n)$ and $J = \mathbb{Z}_+$. We define the *scaled replicas* of a structuring element $B \subseteq \mathbb{Z}^n$ by

$$\sigma B = \begin{cases} \underbrace{B \oplus B \oplus \cdots \oplus B}_{\sigma - 1 \text{ times}}, & \text{for } \sigma \in \mathbb{Z}_+, \sigma \neq 0 \\ \mathbf{0}, & \text{for } \sigma = 0 \end{cases}, \tag{6.31}$$

where $\mathbf{0}$ is the origin of $\mathbb{Z}^n$. Consider the family $\{\delta_\sigma(A) = A \oplus \sigma B \mid \sigma \in \mathbb{Z}_+\}$ of discrete dilations, where B is *any* structuring element. From [34, Prop. 4.10], it follows that, if $\sigma \geq \tau$, then $\delta_\sigma(A) = A \oplus \sigma B = (A \oplus \tau B) \oplus (\sigma - \tau)B = \delta_{\sigma-\tau}\delta_\tau(A)$, so that (6.29) is satisfied, and $\{\delta_\sigma(A) = A \oplus \sigma B \mid \sigma \in \mathbb{Z}_+\}$ has the interpolation property.

(c) (*Granulometry*). Let $\mathcal{L}$ be an arbitrary lattice and J be an arbitrary poset. Consider a granulometry $\{\theta_\sigma \mid \sigma \in J\}$ (see Section 2.2). From Proposition 2.2.9, it follows that, if $\sigma \geq \tau$, then $\theta_\sigma = \theta_\sigma\theta_\tau$, so that (6.29) is satisfied, and $\{\theta_\sigma \mid \sigma \in J\}$ has the interpolation property.

(d) (*Anti-granulometry*). This is the dual of the previous example. Let $\mathcal{L}$ be an arbitrary lattice and J be an arbitrary poset. Consider an anti-granulometry $\{\phi_\sigma \mid \sigma \in J\}$; i.e., a family $\{\phi_\sigma \mid \sigma \in J\}$ of closings on $\mathcal{L}$ such that $\phi_\sigma \geq \phi_\tau$, for $\sigma \geq \tau$. By a dual statement to the one of Proposition 2.2.9, it follows that, if $\sigma \geq \tau$, then $\phi_\sigma = \phi_\sigma\phi_\tau$, so that (6.29) is satisfied, and $\{\phi_\sigma \mid \sigma \in J\}$ has the interpolation property. $\diamondsuit$

We remark here that Examples 6.3.2(a), (b) can also be defined on a proper subset E of $\mathbb{R}^n$ or $\mathbb{Z}^n$, respectively (this is convenient at times, both from a theoretical and a practical point of view). In this case, the dilations are given by $\delta_\sigma(A) = A \oplus_E \sigma B = (A \oplus \sigma B) \cap E$, for $\sigma \in \mathbb{R}_+$ or $\sigma \in \mathbb{Z}_+$, respectively (see (2.22) and (2.23)).

Now, recall from Section 4.3.1 that clusterings are operators that generate new connectivities by grouping connected components into "clusters." The following definition extends the notion of clustering to a multiscale framework.

6.3.3 Definition. Let $\mathcal{L}$ be a lattice, furnished with a connectivity class $\mathcal{C}$. A family $\{\psi_\sigma \mid \sigma \in \mathbb{R}_+\}$ of operators on $\mathcal{L}$ is said to be a *clustering pyramid* on $\mathcal{L}$ if:

(*i*) ψ_σ is a clustering on $\mathcal{L}$, for each $\sigma \in \mathbb{R}_+$,

(*ii*) $\{\psi_\sigma \mid \sigma \in \mathbb{R}_+\}$ has the interpolation property,

(*iii*) $\psi_\tau(A) \in \mathcal{C}$, for all $\tau > \sigma \;\Rightarrow\; \psi_\sigma(A) \in \mathcal{C}$, for $\sigma \in \mathbb{R}_+$, $A \in \mathcal{L}$.

A clustering pyramid is said to be *strong* if each clustering in the pyramid is strong. $\triangle$

Given a clustering pyramid $\{\psi_\sigma \mid \sigma \in \mathbb{R}_+\}$ on $\mathcal{L}$, we say that $A \in \mathcal{L}$ is a σ-*cluster* if $\psi_\sigma(A) \in \mathcal{C}$, for $\sigma \in \mathbb{R}_+$. Properties (*i*) and (*ii*) of Definition 6.3.3 are self-evident, whereas property (*iii*) says that if A is a τ-cluster for each $\tau > \sigma$, then A must be a σ-cluster, for $\sigma \in \mathbb{R}_+$. This provides a smoothness constraint on the clustering pyramid.

Clustering pyramids lead to multiscale connectivities. This is shown by the following proposition.

6.3.4 Proposition. Let $\mathcal{L}$ be a lattice, furnished with a connectivity class $\mathcal{C}$. If $\{\psi_\sigma \mid \sigma \in \mathbb{R}_+\}$ is a clustering pyramid on $\mathcal{L}$, then $\mathbf{C}$: $\mathbb{R} \to \mathcal{P}(\mathcal{L})$, given by

$$\mathbf{C}(\sigma) = \begin{cases} \mathcal{C}, & \text{if } \sigma > 0 \\ (\psi_{-\sigma})^{-1}(\mathcal{C}) = \{A \in \mathcal{L} \mid \psi_{-\sigma}(A) \in \mathcal{C}\}, & \text{if } \sigma \leq 0 \end{cases}, \quad \sigma \in \mathbb{R}, \tag{6.32}$$

is a connectivity pyramid on $\mathcal{L}$, with associated connectivity measure φ: $\mathcal{L} \to \overline{\mathbb{R}}$ given by

$$\varphi(A) = \begin{cases} \infty, & \text{if } A \in \mathcal{C} \\ -\bigwedge\{\sigma \in \mathbb{R}_+ \mid \psi_\sigma(A) \in \mathcal{C}\}, & \text{if } A \notin \mathcal{C} \end{cases}, \quad A \in \mathcal{L}. \tag{6.33}$$

□

Proof. From Proposition 4.3.5(a), it follows that $\mathbf{C}(\sigma)$ is a connectivity class in $\mathcal{L}$, for each $\sigma \in \mathbb{R}$, which shows axiom (i) of a connectivity pyramid. To show axiom (ii), we need to prove that $A \in \mathbf{C}(\sigma) \Rightarrow A \in \mathbf{C}(\tau)$, if $\sigma \geq \tau$. If $\sigma, \tau > 0$, there is nothing to show. If $\sigma > 0$ and $\tau \leq 0$, then $A \in \mathbf{C}(\sigma) = \mathcal{C}$, so that $\psi_{-\tau}(A) \in \mathcal{C} \Rightarrow A \in \mathbf{C}(\tau)$, since $\psi_{-\tau}$ is connectivity-preserving. If $\sigma, \tau \leq 0$, we have that $A \in \mathbf{C}(\sigma) = (\psi_{-\sigma})^{-1}(\mathcal{C}) \Rightarrow \psi_{-\sigma}(A) \in \mathcal{C}$. From the interpolation property of the clustering pyramid, there exists a $\tau' \in \mathbb{R}_+$ such that $\psi_{-\tau}(A) = \psi_{\tau'}\psi_{-\sigma}(A)$. But since $\psi_{\tau'}$ is connectivity-preserving, it follows that $\psi_{-\tau}(A) \in \mathcal{C} \Rightarrow A \in \mathbf{C}(\tau)$. To show axiom ($iii$), we need to prove that $A \in \mathbf{C}(\sigma) \Leftrightarrow A \in \mathbf{C}(\tau)$, for all $\tau < \sigma$. The direct implication follows from axiom (ii). To show the converse implication, note that if $\sigma > 0$, there is nothing to show, whereas if $\sigma \leq 0$, the desired result follows easily from property (iii) of Definition 6.3.3.

Finally, we verify (6.33). From (6.11), we have that $\varphi(A) = \bigvee\{\sigma \in \mathbb{R} \mid A \in \mathbf{C}(\sigma)\}$, for $A \in \mathcal{L}$. If $A \in \mathcal{C}$, then $A \in \mathbf{C}(\sigma)$, for all $\sigma \in \mathbb{R}$, so that $\varphi(A) = \bigvee \mathbb{R} = \infty$. If $A \notin \mathcal{C}$ (i.e., if $A \notin \mathbf{C}(\sigma)$, for $\sigma > 0$), then $\varphi(A) = \bigvee\{\sigma \in \mathbb{R}_- \mid A \in \mathbf{C}(\sigma)\} = \bigvee\{\sigma \in \mathbb{R}_- \mid \psi_{-\sigma}(A) \in \mathcal{C}\} = -\bigwedge\{\sigma \in \mathbb{R}_+ \mid \psi_\sigma(A) \in \mathcal{C}\}$, as required, where $\mathbb{R}_-$ denotes the set of nonpositive real numbers. Q.E.D.

The multiscale connectivity defined above is referred to as a *clustering-pyramid multiscale connectivity*. In this framework, if $A \in \mathcal{C}$, then A is fully connected, whereas if $A \notin \mathcal{C}$, then its degree of connectivity is negative (nonpositive), in which case A is $-\sigma$-connected if $\psi_{-\sigma}(A) \in \mathcal{C}$, for $\sigma \in \mathbb{R}_+$. Hence, connectivity at negative scales corresponds to how disconnected A is with respect to the base connectivity $\mathcal{C}$; the more negative the degree of connectivity of A is, the "larger" the clustering applied on A needs to be in order to "reconnect" A. Note that $\psi_\sigma(A)$ may be thought of as A seen at scale $-\sigma$, so that $-\sigma$-connectivity of A corresponds to connectivity, according to $\mathcal{C}$, of A at scale $-\sigma$, for $\sigma \in \mathbb{R}_+$. Note also that A is fully disconnected if $\psi_{-\sigma}(A) \notin \mathcal{C}$, for all $\sigma \in \mathbb{R}_+$; i.e., no clustering in the pyramid is enough to reconnect A.

As a straightforward consequence of Proposition 4.3.6, it is possible to characterize the σ-connectivity openings and the σ-reconstruction operators associated with a multiscale connectivity generated by a strong clustering pyramid. We have the following proposition.

6.3.5 Proposition. Let $\mathcal{L}$ be a lattice with a sup-generating family $\mathcal{S}$, furnished with a connectivity class $\mathcal{C}$. Let $(\varphi, \mathbf{C})$ be the multiscale connectivity system generated by a strong clustering pyramid $\{\psi_\sigma \mid \sigma \in \mathbb{R}_+\}$ on $\mathcal{L}$, given by (6.32) and (6.33). Let $\{\gamma_x \mid x \in \mathcal{S}\}$ and ρ be the connectivity openings and the reconstruction operator, respectively, associated with $\mathcal{C}$. Then:

(a) The σ-connectivity openings associated with $(\varphi, \mathbf{C})$ are given by

$$\gamma_{\sigma,x}(A) = \begin{cases} A \wedge \gamma_x \psi_{-\sigma}(A), & \text{if } \ x \leq A \\ O, & \text{if } \ x \not\leq A \end{cases}, \quad \sigma \leq 0, \tag{6.34}$$

and

$$\gamma_{\sigma,x}(A) = \gamma_x(A), \quad \sigma > 0, \tag{6.35}$$

for $A \in \mathcal{L}$, $x \in \mathcal{S}$.

(b) If $\mathcal{L}$ is infinite $\vee$-distributive, the σ-reconstruction operators associated with $(\varphi, \mathbf{C})$ are given by

$$\rho_\sigma(A \mid M) = \begin{cases} \rho(A|M), & \text{for } \ \sigma > 0 \\ A \wedge \rho(\psi_{-\sigma}(A) \mid A \wedge M), & \text{for } \ \sigma \leq 0 \end{cases}, \tag{6.36}$$

for $A, M \in \mathcal{L}$. □

The following corollary is an immediate consequence of part (a) of the previous proposition.

6.3.6 Corollary. Let $\mathcal{L}$ be a lattice with a sup-generating family $\mathcal{S}$, furnished with a connectivity class $\mathcal{C}$, with $\downarrow$-continuous connectivity openings. Let $(\varphi, \mathbf{C})$ be the multiscale connectivity system generated by a strong clustering pyramid $\{\psi_\sigma \mid \sigma \in \mathbb{R}_+\}$ on $\mathcal{L}$, given by (6.32) and (6.33). If, for each $\sigma \in \mathbb{R}_+$, the clustering ψ_σ is $\downarrow$-continuous, then the σ-connectivity openings $\{\gamma_{\sigma,x} \mid \sigma \in \mathbb{R}, x \in \mathcal{S}\}$ associated with $(\varphi, \mathbf{C})$ are $\downarrow$-continuous. $\square$

Before we provide examples, we need a few auxiliary results. In particular, we are interested in studying properties of the lattice $\mathcal{F}(E)$ of closed sets of a topological space E.

6.3.7 Lemma. Let E be a topological space and J be a poset. Suppose that $\{\widehat{\psi}_\alpha \mid \alpha \in J\}$ is a family of operators on $\mathcal{P}(E)$ such that the restriction of $\widehat{\psi}_\alpha$ to $\mathcal{F}(E)$ defines an operator ψ_α on $\mathcal{F}(E)$, for $\alpha \in J$ (i.e., $A \in \mathcal{F}(E) \Rightarrow \widehat{\psi}_\alpha(A) \in \mathcal{F}(E)$, for $\alpha \in J$).

(a) If the family $\{\widehat{\psi}_\alpha \mid \alpha \in J\}$ has the interpolation property, so does the family $\{\psi_\alpha \mid \alpha \in J\}$.

(b) Suppose that E is a Hausdorff space and $\widehat{\mathcal{C}}$ is a compatible connectivity class in $\mathcal{P}(E)$, with the points as sup-generators. If $\{\widehat{\psi}_\alpha \mid \alpha \in J\}$ is a family of strong clusterings on $\mathcal{P}(E)$, according to $\widehat{\mathcal{C}}$, then $\{\psi_\alpha \mid \alpha \in J\}$ is a family of strong clusterings on $\mathcal{F}(E)$, according to the connectivity class $\mathcal{C} = \widehat{\mathcal{C}} \cap \mathcal{F}(E)$. $\square$

PROOF. (a): For $\alpha, \beta \in J$, with $\alpha \geq \beta$, there exists a $\gamma \in J$ such that, for all $A \in \mathcal{F}(E)$, $\psi_\alpha(A) = \widehat{\psi}_\alpha(A) = \widehat{\psi}_\gamma \widehat{\psi}_\beta(A) = \psi_\gamma \psi_\beta(A)$; i.e., $\psi_\alpha = \psi_\gamma \psi_\beta$, as required.

(b): Let $\alpha \in J$. We show that ψ_α satisfies the conditions of Definition 4.3.2. Clearly, ψ_α is increasing and anti-extensive on $\mathcal{F}(E)$, and $\psi_\alpha(O) = \widehat{\psi}_\alpha(O) = O$. Now, from Proposition 4.1.11, it follows that the connectivity openings $\{\gamma_x \mid x \in \mathcal{S}\}$ associated with $\mathcal{C}$ are the restriction to $\mathcal{F}(E)$ of the connectivity openings $\{\widehat{\gamma}_x \mid x \in \mathcal{S}\}$ associated with $\widehat{\mathcal{C}}$. Therefore, for all $A \in \mathcal{F}(E)$ and $x \in \mathcal{S}$, $\psi_\alpha(A \cap \gamma_x \psi_\alpha(A)) = \widehat{\psi}_\alpha(A \cap \widehat{\gamma}_x \widehat{\psi}_\alpha(A)) = \widehat{\gamma}_x \widehat{\psi}_\alpha(A) = \gamma_x \psi_\alpha(A)$; i.e., $\psi_\alpha(A \wedge \gamma_x \psi_\alpha) = \gamma_x \psi_\alpha$, as required. Q.E.D.

The previous result gives sufficient conditions so that $\{\psi_\alpha \mid \alpha \in J\}$ is a family of clusterings on $\mathcal{F}(E)$ that has the interpolation property. Such families satisfy properties (i) and (ii) of a clustering pyramid. The following result gives a sufficient condition that guarantees that property (iii) is also satisfied.

6.3.8 Lemma. Let $\mathcal{L}$ be a lattice with a sup-generating family $\mathcal{S}$, furnished with a connectivity class $\mathcal{C}$, with $\downarrow$-continuous connectivity openings $\{\gamma_x \mid x \in \mathcal{S}\}$. A family $\{\psi_\sigma \mid \sigma \in \mathbb{R}_+\}$

of clusterings on $\mathcal{L}$ that has the interpolation property, and such that

$$\psi_\sigma = \bigwedge_{\tau>\sigma} \psi_\tau, \quad \sigma \in \mathbb{R}_+, \tag{6.37}$$

is a clustering pyramid on $\mathcal{L}$. □

PROOF. We need to show property (*iii*) of Definition 6.3.3. Given $\sigma \in \mathbb{R}_+$ and $A \in \mathcal{L}$, suppose that $\psi_\tau(A) \in \mathcal{C}$, for all $\tau > \sigma$. If $\psi_\sigma(A) = O$, then $\psi_\sigma(A) \in \mathcal{C}$ and we are done. Otherwise, pick a sup-generator $x \leq A$. Clearly, $\{\psi_\tau(A) \mid \tau > \sigma\}$ is a decreasing family in $\mathcal{L}$. Since γ_x is $\downarrow$-continuous, we can apply Proposition 2.2.10 to get $\gamma_x\psi_\sigma(A) = \gamma_x(\bigwedge_{\tau>\sigma} \psi_\tau(A)) = \bigwedge_{\tau>\sigma} \gamma_x\psi_\tau(A) = \bigwedge_{\tau>\sigma} \psi_\tau(A) = \psi_\sigma(A)$. Therefore, $\psi_\sigma(A) \in \mathcal{C}$, as required. Q.E.D.

The two previous lemmas are useful for establishing the next result, which gives an important example of a clustering pyramid.

6.3.9 Proposition. Let E be a connected, bounded and closed subset of $\mathbb{R}^n$, with the Euclidean topology. Let $\mathcal{L} = \mathcal{F}(E)$, with the points as sup-generators, furnished with the connectivity class $\mathcal{C}$ of topologically connected closed sets in E, and let $B \subseteq \mathbb{R}^n$ be a compact structuring element.

(a) The restriction of the dilation $\widehat{\delta}_\sigma(A) = A \oplus_E \sigma B$ on $\mathcal{P}(E)$ to $\mathcal{F}(E)$ defines an operator δ_σ on $\mathcal{F}(E)$; i.e.,

$$A \in \mathcal{F}(E) \Rightarrow A \oplus_E \sigma B \in \mathcal{F}(E), \quad \sigma \in \mathbb{R}_+. \tag{6.38}$$

(b) We have that $\delta_\sigma = \bigwedge_{\tau>\sigma} \delta_\tau$, for $\sigma \in \mathbb{R}_+$; i.e., for $A \in \mathcal{F}(E)$,

$$A \oplus_E \sigma B = \bigcap_{\tau>\sigma} (A \oplus_E \tau B), \quad \sigma \in \mathbb{R}_+. \tag{6.39}$$

(c) If B is convex, B contains the origin of $\mathbb{R}^n$, and $(\sigma B)_v \cap E$ is connected, for all $v \in E$ and $\sigma \in \mathbb{R}_+$, then the family $\{\delta_\sigma(A) \mid \sigma \in \mathbb{R}_+\}$ is a strong clustering pyramid on $\mathcal{F}(E)$. □

PROOF. (a): Compactness of B in $\mathbb{R}^n$ implies that σB is also compact in $\mathbb{R}^n$, for $\sigma \in \mathbb{R}_+$. Since A is closed in $\mathbb{R}^n$, it follows from [34, Lemma 7.42] that $A \oplus \sigma B$ is closed in $\mathbb{R}^n$, so that $A \oplus_E \sigma B = (A \oplus \sigma B) \cap E \in \mathcal{F}(E)$, for $\sigma \in \mathbb{R}_+$.

(b): The inequality $A \oplus_E \sigma B \subseteq \bigcap_{\tau>\sigma} A \oplus_E \tau B$, for $\sigma \in \mathbb{R}_+$, is obvious. We show the reverse inequality. Let $\sigma \in \mathbb{R}_+$, and suppose that $v \notin A \oplus \sigma B$; i.e., $v \in (A \oplus \sigma B)^c$. Note that this is equivalent to the fact that there is no $r \in A$ and $s \in B$ such that $v = r + \sigma s$. Since $(A \oplus \sigma B)^c$ is an open set, we can find an open Euclidean ball $B(v, \epsilon)$, centered at v and with radius $\epsilon > 0$, such that $B(v, \epsilon) \subseteq (A \oplus \sigma B)^c$. Clearly, this implies that there is no $r \in A$ and $s \in B$ such that $v = r + (\sigma + \epsilon/2)s$; i.e., $v \notin A \oplus (\sigma + \epsilon/2)B$, so that $v \notin \bigcap_{\tau>\sigma}(A \oplus \tau B)$. Hence, $A \oplus \sigma B \supseteq \bigcap_{\tau>\sigma} A \oplus \tau B \Rightarrow A \oplus_E \sigma B = (A \oplus \sigma B) \cap E \supseteq (\bigcap_{\tau>\sigma} A \oplus \tau B) \cap E = \bigcap_{\tau>\sigma}((A \oplus \tau B) \cap E) = \bigcap_{\tau>\sigma} A \oplus_E \tau B$, as required.

(c): Convexity of B implies that $\{\widehat{\delta}_\sigma \mid \sigma \in \mathbb{R}_+\}$ has the interpolation property (see Example 6.3.2(a)). It follows from Lemma 6.3.7(a) that $\{\delta_\sigma \mid \sigma \in \mathbb{R}_+\}$ has the interpolation property as well. In addition, since B contains the origin of $\mathbb{R}^n$, the dilation $\widehat{\delta}_\sigma$ is extensive, for each $\sigma \in \mathbb{R}_+$. Moreover, we have that $\widehat{\delta}_\sigma(\{v\}) = \{v\} \oplus_E \sigma B = (\sigma B)_v \cap E$ is connected, for all $v \in E$ and $\sigma \in \mathbb{R}_+$. Since $\mathcal{P}(E)$ is an infinite $\vee$-distributive lattice, we can apply Proposition 4.3.9 to conclude that $\{\widehat{\delta}_\sigma \mid \sigma \in \mathbb{R}_+\}$ is a family of strong clusterings on $\mathcal{P}(E)$, according to the connectivity class $\widehat{\mathcal{C}}$ of the topologically connected sets in E. Note that $\mathcal{C} = \widehat{\mathcal{C}} \cap \mathcal{F}(E)$. It then follows from Lemma 6.3.7(b) that $\{\delta_\sigma(A) \mid \sigma \in \mathbb{R}_+\}$ is a family of strong clusterings on $\mathcal{F}(E)$, according to $\mathcal{C}$. Finally, note that E is a connected compact Hausdorff space. It follows from Proposition 4.1.13 that the connectivity openings associated with $\mathcal{C}$ are $\downarrow$-continuous. In addition, from part (b), we have that $\delta_\sigma = \bigwedge_{\tau>\sigma} \delta_\tau$, for $\sigma \in \mathbb{R}_+$. Therefore, we can apply Lemma 6.3.8 to conclude that $\{\delta_\sigma(A) \mid \sigma \in \mathbb{R}_+\}$ is a clustering pyramid on $\mathcal{F}(E)$. Q.E.D.

The following result follows easily from Proposition 7.39 and Corollary 7.44, in [34].

6.3.10 Proposition. Let E be a bounded and closed subset of $\mathbb{R}^n$ with the Euclidean topology. For $B \in \mathcal{F}(E)$, the operator $\delta_\sigma(A) = A \oplus_E \sigma B$ on $\mathcal{F}(E)$ is $\downarrow$-continuous, for $\sigma \in \mathbb{R}_+$. □

Propositions 6.3.4–6.3.10 and Corollary 6.3.6 lead to the following example.

6.3.11 Example. Let E be a connected, bounded and closed subset of $\mathbb{R}^n$, with the Euclidean topology. Let $\mathcal{L} = \mathcal{F}(E)$, with the points as sup-generators, furnished with the connectivity class $\mathcal{C}$ of topologically connected closed sets in E, and let $B \subseteq \mathbb{R}^n$ be a compact structuring element, such that B is convex, B contains the origin of $\mathbb{R}^n$, and $(\sigma B)_v \cap E$ is connected, for all $v \in E$ and $\sigma \in \mathbb{R}_+$. The family $\{\delta_\sigma(A) = A \oplus_E \sigma B \mid \sigma \in \mathbb{R}_+\}$ is a

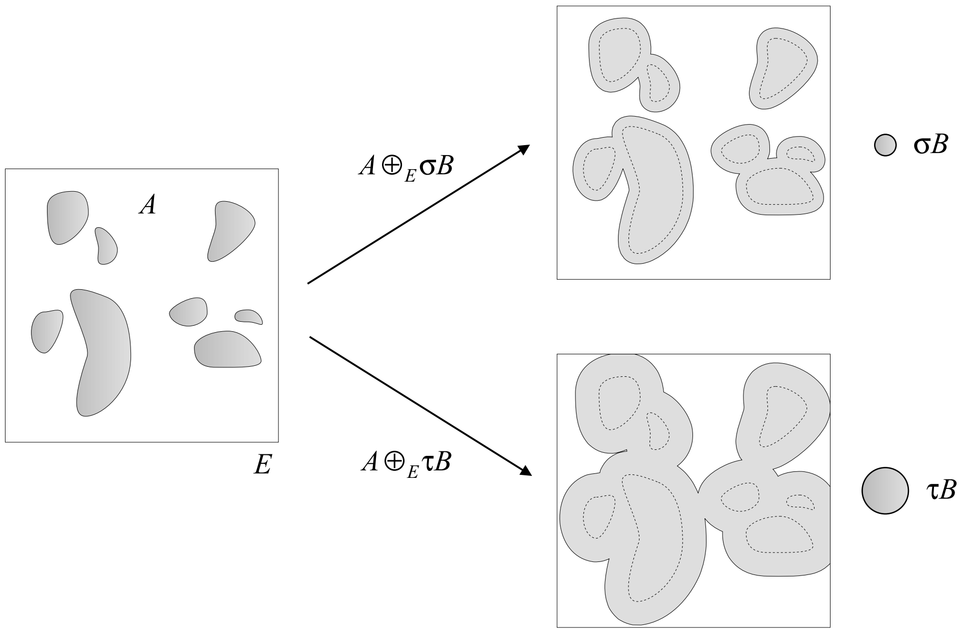


Figure 6.2: An illustration of dilation-pyramid multiscale connectivity. The original set A is disconnected; hence, its degree of connectivity $\varphi(A)$ is negative. Note that $A \oplus_E \tau B$ is connected, but $A \oplus_E \sigma B$ is not; hence, A is $-\tau$-connected, but it is not $-\sigma$-connected. Equivalently, $-\tau \leq \varphi(A) < -\sigma$.

strong clustering pyramid on $\mathcal{F}(E)$, with associated *dilation-pyramid* multiscale connectivity system $(\varphi, \mathbf{C})$, given by (6.32) and (6.33), with $\psi_\sigma = \delta_\sigma$, for $\sigma \in \mathbb{R}_+$. Furthermore, the σ-connectivity openings $\{\gamma_{\sigma,x} \mid \sigma \in \mathbb{R}, x \in \mathcal{S}\}$ associated with $(\varphi, \mathbf{C})$ are $\downarrow$-continuous (we remark that, despite the term "dilation-pyramid," the operators $\{\delta_\sigma \mid \sigma \in \mathbb{R}_+\}$ are pseudo-dilations on $\mathcal{F}(E)$, since there are instances in which they fail to commute with the supremum in $\mathcal{F}(E)$). $\Diamond$

Note that, in this framework, we have that $\varphi(A) = \infty$, if $A \in \mathcal{C}$; otherwise, $\varphi(A) = -\bigwedge\{\sigma \in \mathbb{R}_+ \mid A \oplus_E \sigma B \in \mathcal{C}\}$; i.e., the negative of the "infimum size" of dilation by B necessary to obtain a connected set. Fig. 6.2 provides an illustration of this example, where E is a square subset of $\mathbb{R}^2$, furnished with the Euclidean topology, and $B \in \mathcal{F}(E)$ is a Euclidean disk. Note that the original set A is disconnected, so that it has a negative degree of connectivity. Dilating A by increasingly larger scaled replicas of the structuring element B eventually produces a connected set. The (negative) degree of connectivity of A measures how disconnected A is, with respect to the base connectivity $\mathcal{C}$.

We mention here that multiscale connectivities can also be generated by certain anti-granulometries of connectivity-preserving closings; this is a consequence of Example 6.3.2(d) and Lemma 6.3.8. We have the following example.

6.3.12 Example. Let $\mathcal{L}$ be an arbitrary lattice, furnished with a connectivity class $\mathcal{C}$, with $\downarrow$-continuous connectivity openings $\{\gamma_x \mid x \in \mathcal{S}\}$. An anti-granulometry $\{\phi_\sigma \mid \sigma \in \mathbb{R}_+\}$ of connectivity-preserving closings on $\mathcal{L}$, such that $\phi_\sigma = \bigwedge_{\tau > \sigma} \phi_\tau$, for each $\sigma \in \mathbb{R}_+$, is a clustering pyramid on $\mathcal{L}$. The associated *closing-pyramid* multiscale connectivity system $(\varphi, \mathbf{C})$ is given by (6.32) and (6.33), with $\psi_\sigma = \phi_\sigma$, for $\sigma \in \mathbb{R}_+$. $\Diamond$

The discretization of the theory of clustering-pyramid multiscale connectivity presented above is straightforward. The following is the discrete analog of Definition 6.3.3.

6.3.13 Definition. Let $\mathcal{L}$ be a lattice, furnished with a connectivity class $\mathcal{C}$. A family $\{\psi_\sigma \mid \sigma \in \mathbb{Z}_+\}$ of clusterings on $\mathcal{L}$ that has the interpolation property is said to be a *discrete clustering pyramid* on $\mathcal{L}$. A discrete clustering pyramid is said to be *strong* if each clustering in the pyramid is strong. $\triangle$

The following is the discrete analog of Proposition 6.3.4.

6.3.14 Proposition. Let $\mathcal{L}$ be a lattice, furnished with a connectivity class $\mathcal{C}$. If $\{\psi_\sigma \mid \sigma \in \mathbb{Z}_+\}$ is a discrete clustering pyramid on $\mathcal{L}$, then $\mathbf{C}$: $\mathbb{Z} \to \mathcal{P}(\mathcal{L})$, given by

$$\mathbf{C}(\sigma) = \begin{cases} \mathcal{C}, & \text{if } \sigma > 0 \\ (\psi_{-\sigma})^{-1}(\mathcal{C}) = \{A \in \mathcal{L} \mid \psi_{-\sigma}(A) \in \mathcal{C}\}, & \text{if } \sigma \leq 0 \end{cases}, \quad \sigma \in \mathbb{Z}, \tag{6.40}$$

is a discrete connectivity pyramid on $\mathcal{L}$, with associated discrete connectivity measure φ: $\mathcal{L} \to \overline{\mathbb{Z}}$ given by

$$\varphi(A) = \begin{cases} \infty, & \text{if } A \in \mathcal{C} \\ -\bigwedge\{\sigma \in \mathbb{Z}_+ \mid \psi_\sigma(A) \in \mathcal{C}\}, & \text{if } A \notin \mathcal{C} \end{cases}, \quad A \in \mathcal{L}. \tag{6.41}$$

$\square$

The discrete multiscale connectivity defined above is referred to as a *discrete clustering-pyramid multiscale connectivity*. All remarks made previously regarding clustering-pyramid multiscale connectivities apply to the discrete case as well, with obvious modifications. In particular, the discrete analog of Proposition 6.3.5 is straightforward and will not be repeated here.

The following example follows easily from Proposition 4.3.9, Proposition 6.3.14 and Examples 6.3.2(a),(b).

6.3.15 Example. Let E be a (not necessarily proper) subset of $\mathbb{R}^n$ or $\mathbb{Z}^n$, and let $\mathcal{L} = \mathcal{P}(E)$, with the points as sup-generators, furnished with a connectivity class $\mathcal{C}$. If B is convex (in the case in which $E \subseteq \mathbb{R}^n$), B contains the origin of $\mathbb{R}^n$ or $\mathbb{Z}^n$, and $(\sigma B)_v \cap E$ is connected, for all $v \in E$ and $\sigma \in \mathbb{Z}_+$, then the family $\{\delta_\sigma(A) = A \oplus_E \sigma B \mid \sigma \in \mathbb{Z}_+\}$ is a strong discrete clustering pyramid on $\mathcal{P}(E)$. The associated discrete dilation-pyramid multiscale connectivity system $(\varphi, \mathbf{C})$ is given by (6.40) and (6.41), with $\psi_\sigma = \delta_\sigma$, for $\sigma \in \mathbb{Z}_+$. $\Diamond$

Note that, in this framework, we have that $\varphi(A) = \infty$, if $A \in \mathcal{C}$; otherwise, $\varphi(A) = -m$, where m is the minimum number of dilations by B necessary to obtain a connected set. This provides a practical algorithm for computing $\varphi(A)$. Fig. 6.3 provides an illustration of this example, where E is a rectangle in $\mathbb{Z}^2$, the base connectivity is given by the 4-adjacency connectivity and the basic structuring element is the 3×3 cross. This figure depicts a 256-graylevel image of cornea cells, which was pre-processed by an alternating sequential filter [77] in order to reduce noise. Thresholding this image with increasing threshold values produces the three discrete binary images A_1, A_2, and A_3 depicted in Fig. 6.3. The associated degrees of connectivity are also displayed. Since all three images are disconnected, their degree of connectivity is negative. Note that the more negative the degree of connectivity is, the more spread apart the particles that make up the images are.

Clearly, a discrete multiscale connectivity can be generated by an anti-granulometry of connectivity-preserving closings as well; this is a consequence of Example 6.3.2(d). We have the following example.

6.3.16 Example. Let $\mathcal{L}$ be an arbitrary lattice, furnished with a connectivity class $\mathcal{C}$. An anti-granulometry $\{\phi_\sigma \mid \sigma \in \mathbb{R}_+\}$ of connectivity-preserving closings on $\mathcal{L}$ is a discrete clustering pyramid on $\mathcal{L}$. The associated discrete closing-pyramid multiscale connectivity system $(\varphi, \mathbf{C})$ is given by (6.40) and (6.41), with $\psi_\sigma = \phi_\sigma$, for $\sigma \in \mathbb{R}_+$. $\Diamond$

An interesting example of discrete closing-pyramid multiscale connectivity, based on morphological sampling operators, was originally suggested to us by H. Heijmans (personal communication). In Section 4.3.1, a brief review of the theory of morphological sampling was provided. In what follows, we consider the case in which $S = \{k_1 u_1 + \cdots + k_n u_n \mid k_i \in \mathbb{Z}\}$ and $C = \{x_1 u_1 + \cdots + x_n u_n \mid -1 < x_i < 1\}$, for linearly independent vectors $\{u_i \mid i = 1, 2, \ldots, n\}$

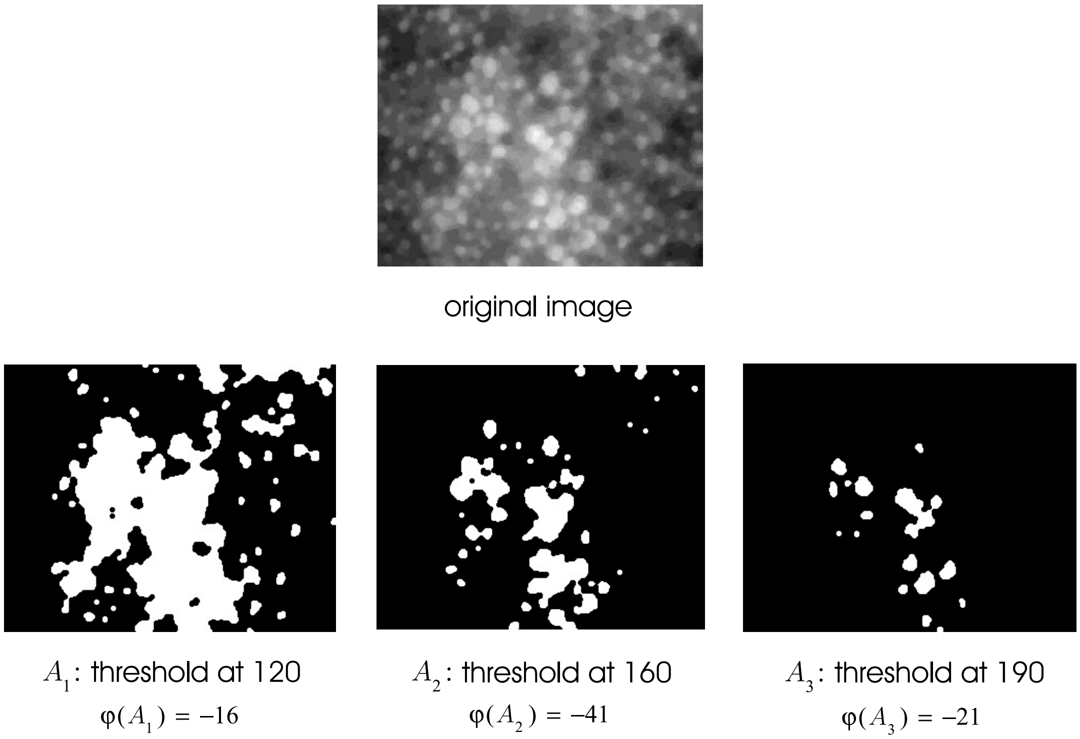


Figure 6.3: An illustration of discrete dilation-pyramid multiscale connectivity. The original 256-graylevel cornea cell image is thresholded to produce the three discrete binary images A_1, A_2, and A_3. Since the images are disconnected, their degrees of connectivity are negative. The more negative the degree of connectivity is, the more spread apart the particles that make up the images are.

in $\mathbb{R}^n$ (in the special case in which $\{u_i \mid i = 1, 2, \ldots, n\}$ is the orthonormal basis of $\mathbb{R}^n$, this scheme corresponds to the most commonly available discretization architecture implemented in hardware).

Let $\{\pi_\sigma \mid \sigma \in \mathbb{Z}_+\}$ be a family of approximation closings, given by

$$\pi_\sigma(A) = \left(\bigcup_{s \in S_\sigma} \{(C_\sigma)_s \mid (C_\sigma)_s \cap A = \emptyset\} \right)^c, \quad A \in \mathcal{P}(\mathbb{R}^n), \tag{6.42}$$

where $\{S_\sigma \mid \sigma \in \mathbb{Z}_+\}$ and $\{C_\sigma \mid \sigma \in \mathbb{Z}_+\}$ are families of increasingly coarser sampling grids and sampling elements, respectively, given by

$$S_\sigma = 2^\sigma S, \quad C_\sigma = 2^\sigma C, \quad \sigma \in \mathbb{Z}_+. \tag{6.43}$$

Given a set $A \in \mathcal{P}(\mathbb{R}^n)$, $\pi_\sigma(A)$ gives the discretization of A at scale σ. Note that $S_0 = S$ and $C_0 = C$. The *basic discretization* $\pi_0(A)$ gives the finest possible discretization of A.

This may be interpreted as the finest degree of scale allowed by a particular discretization device. We have the following result.

6.3.17 Proposition. The family $\{\pi_\sigma \mid \sigma \in \mathbb{Z}_+\}$ of approximation closings is an anti-granulometry on $\mathcal{P}(\mathbb{R}^n)$. $\square$

PROOF. We need to show that $\pi_\sigma \geq \pi_\tau$, for $\sigma \geq \tau$. Given $A \in \mathcal{L}$, we show that $\pi_\sigma(A)^c \subseteq \pi_\tau(A)^c$. For convenience, we use the notation $C_\sigma(s) = (C_\sigma)_s$. Let $v \in \pi_\sigma(A)^c$. From (6.42), it follows that this is equivalent to the fact that there is an $s_0 \in S_\sigma$ such that $v \in C_\sigma(s_0) \cap A = \emptyset$. Now, it is easy to verify that $C_\sigma(s) = \bigcup_{s' \in S_\tau} \{C_\tau(s') \mid C_\tau(s') \subseteq C_\sigma(s)\}$, for any $s \in S_\sigma$; i.e., the larger sampling element $C_\sigma(s)$ equals a union of appropriately translated smaller sampling elements $C_\tau(s')$. Therefore, we can find an $s_1 \in S_\tau$ such that $v \in C_\tau(s_1) \subseteq C_\sigma(s_0)$, so that $v \in C_\tau(s_1) \cap A = \emptyset$. In other words, $v \in \pi_\tau(A)^c$, as required. Q.E.D.

We define a connectivity class $\mathcal{C}$ in $\mathcal{P}(\mathbb{R}^n)$ to be *scaling-invariant* if $A \in \mathcal{C} \Leftrightarrow \sigma A \in \mathcal{C}$, for all $\sigma > 0$. For instance, the Euclidean topological connectivity class in $\mathcal{P}(\mathbb{R}^n)$ is scaling-invariant. Combining Proposition 6.3.17 and Proposition 4.3.11 yields the following result.

6.3.18 Proposition. Let $\mathcal{L} = \mathcal{P}(\mathbb{R}^n)$, furnished with a scaling-invariant connectivity class $\mathcal{C}$ such that

$$C_s \setminus (R \oplus C) \in \mathcal{C}, \quad \text{for all } s \in S,\ R \subseteq S. \tag{6.44}$$

Then, the family $\{\pi_\sigma \mid \sigma \in \mathbb{Z}_+\}$ of approximation closings is a discrete clustering pyramid on $\mathcal{P}(\mathbb{R}^n)$. $\square$

Note that condition (6.44) is the same as condition (4.55). As we remarked in Section 4.3.1, this condition is actually easy to check in practice. In particular, if $\mathcal{C}$ is translation-invariant, then (6.44) can be replaced by $C \setminus R \oplus C \in \mathcal{C}$, where R is a combination of nearest neighbors to the origin of S. Since, due to symmetry, some cases are redundant, this results in a small number of tests (for small dimensionality n). For example, it is straightforward to check that the Euclidean topological connectivity class satisfies condition (6.44).

In the framework of the discrete closing-pyramid multiscale connectivity generated by the approximation closings $\{\pi_\sigma \mid \sigma \in \mathbb{Z}_+\}$, σ-connectivity means connectivity at discretization scale $-\sigma$, for $\sigma \leq 0$. We have that $\varphi(A) = \infty$, if $A \in \mathcal{C}$; otherwise, $\varphi(A) = -m$, where

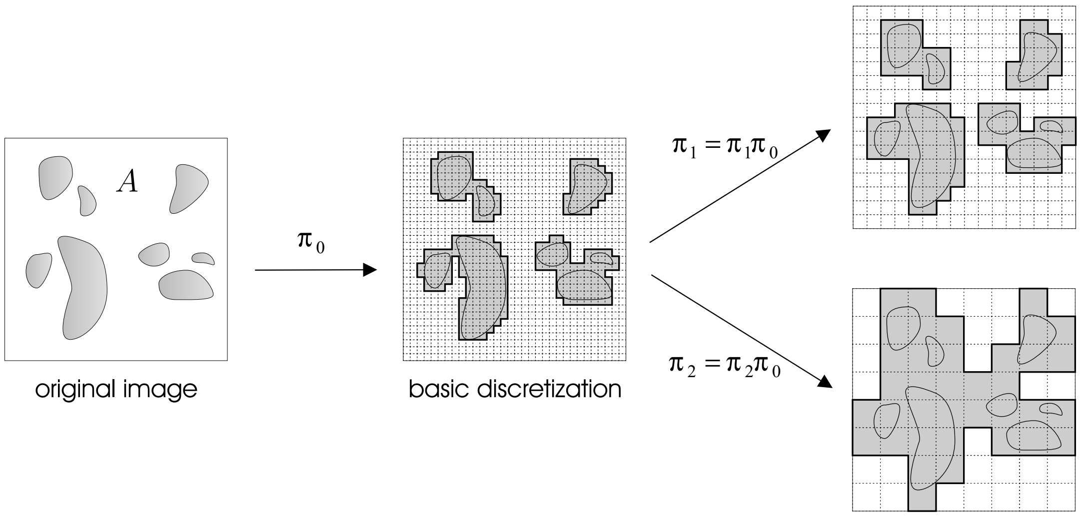


Figure 6.4: An illustration of discrete closing-pyramid multiscale connectivity generated by the approximation closings $\{\pi_\sigma \mid \sigma \in \mathbb{Z}_+\}$. The original image A and its basic discretization $\pi_0(A)$ are disconnected. Hence, the degree of connectivity $\varphi(A)$ is strictly negative. Note that $\pi_2(A)$ is connected, but $\pi_1(A)$ is not; hence, A is -2-connected, but it is not -1-connected. Equivalently, $\varphi(A) = -2$.

m is the minimum discretization scale at which A is connected. Note that when $A \notin \mathcal{C}$, if the basic discretization is the only connected discretization, then $\varphi(A) = 0$; otherwise, $\varphi(A)$ is strictly negative. Fig. 6.4 illustrates this discrete multiscale connectivity framework. Note that the original image A and its basic discretization $\pi_0(A)$ are disconnected, so that A has a strictly negative degree of connectivity. Note also that $\pi_0(A) \subseteq \pi_1(A) \subseteq \pi_2(A)$, as required by the anti-granulometric property of $\{\pi_\sigma \mid \sigma \in \mathbb{Z}_+\}$. Increasingly coarser discretizations of A eventually produce a connected image. The (negative) degree of connectivity of A measures how disconnected A is, with respect to the base connectivity $\mathcal{C}$.

6.3.2 Multiscale Connectivities Generated by Contraction Pyramids

In this subsection, we discuss the notion of contraction pyramids and show that, like clustering pyramids, contraction pyramids can be used to generate multiscale connectivities. We will see that these multiscale connectivities are "positive" in nature.

Recall from Section 4.3.2 that a contraction is any increasing and anti-extensive operator on a lattice $\mathcal{L}$. In the case in which $\mathcal{L}$ is atomic, contractions generate new connectivity classes that consist of the least element, the sup-generators, and the "stable" connected elements. Next, we extend the notion of contraction to a multiscale framework. In what follows, $\mathbb{R}_+^*$ and $\mathbb{Z}_+^*$ denote the set of strictly positive real and strictly positive integer numbers, respectively.

6.3.19 Definition. Let $\mathcal{L}$ be a lattice. A family $\{\xi_\sigma \mid \sigma \in \mathbb{R}_+^*\}$ of operators on $\mathcal{L}$ is said to be a *contraction pyramid* on $\mathcal{L}$ if:

(i) ξ_σ is a contraction on $\mathcal{L}$, for each $\sigma \in \mathbb{R}_+^*$,

(ii) $\xi_\sigma \leq \xi_\tau$, if $\sigma \geq \tau$,

(iii) $\xi_\tau(A) = A$, for all $\tau < \sigma \;\Rightarrow\; \xi_\sigma(A) = A$, for $\sigma \in \mathbb{R}_+^*$, $A \in \mathcal{L}$. △

Given a contraction pyramid $\{\xi_\sigma \mid \sigma \in \mathbb{R}_+^*\}$ on $\mathcal{L}$, we say that $A \in \mathcal{L}$ is σ*-stable* if $\xi_\sigma(A) = A$, for $\sigma \in \mathbb{R}_+^*$. Properties (i) and (ii) of Definition 6.3.19 are self-evident, whereas property (iii) says that if A is τ-stable for each $\tau < \sigma$, then A must be σ-stable, for $\sigma \in \mathbb{R}_+^*$. This provides a smoothness constraint on the contraction pyramid.

Contraction pyramids lead to multiscale connectivities on atomic lattices. This is shown by the following proposition.

6.3.20 Proposition. Let $\mathcal{L}$ be an atomic lattice with sup-generating family $\mathcal{S}$, furnished with a connectivity class $\mathcal{C}$. If $\{\xi_\sigma \mid \sigma \in \mathbb{R}_+^*\}$ is a contraction pyramid on $\mathcal{L}$, then $\mathbf{C}$: $\mathbb{R} \to \mathcal{P}(\mathcal{L})$, given by

$$\mathbf{C}(\sigma) = \begin{cases} \{O\} \cup \mathcal{S} \cup \{A \in \mathcal{C} \mid \xi_\sigma(A) = A\}, & \text{if } \sigma > 0 \\ \mathcal{C}, & \text{if } \sigma \leq 0 \end{cases}, \quad \sigma \in \mathbb{R}, \tag{6.45}$$

is a connectivity pyramid on $\mathcal{L}$, with associated connectivity measure φ: $\mathcal{L} \to \overline{\mathbb{R}}$ given by

$$\varphi(A) = \begin{cases} \infty, & \text{if } A = O \text{ or } A \in \mathcal{S} \\ \bigvee\{\sigma \in \mathbb{R}_+^* \mid \xi_\sigma(A) = A\}, & \text{if } A \in \mathcal{C} \setminus (\{O\} \cup \mathcal{S}) \\ -\infty, & \text{if } A \notin \mathcal{C} \end{cases}, \quad A \in \mathcal{L}. \tag{6.46}$$

□

Proof. For convenience, we use the notation $\overline{\mathcal{S}} = \{O\} \cup \mathcal{S}$. Note that $\mathbf{C}(\sigma) = \overline{\mathcal{S}} \cup (\mathcal{C} \cap \mathrm{Inv}(\xi_\sigma))$, for $\sigma > 0$. From Proposition 4.3.16, it follows that $\mathbf{C}(\sigma)$ is a connectivity class in $\mathcal{L}$, for each $\sigma \in \mathbb{R}$, which shows axiom (i) of a connectivity pyramid. To show axiom (ii), we need to prove that $\mathbf{C}(\sigma) \subseteq \mathbf{C}(\tau)$, if $\sigma \geq \tau$. If $\sigma, \tau \leq 0$, there is nothing to show. If $\sigma \leq 0$ and $\tau > 0$, we have that $\mathbf{C}(\sigma) = \mathcal{C} \subseteq \overline{\mathcal{S}} \cup (\mathcal{C} \cap \mathrm{Inv}(\xi_\sigma)) = \mathbf{C}(\tau)$. If $\sigma, \tau > 0$, we have that

$\xi_\sigma \leq \xi_\tau \Rightarrow \mathrm{Inv}(\xi_\sigma) \subseteq \mathrm{Inv}(\xi_\tau) \Rightarrow \mathbf{C}(\sigma) = \overline{\mathcal{S}} \cup (\mathcal{C} \cap \mathrm{Inv}(\xi_\sigma)) \subseteq \overline{\mathcal{S}} \cup (\mathcal{C} \cap \mathrm{Inv}(\xi_\tau)) = \mathbf{C}(\tau)$. To show axiom (*iii*), we need to prove that $\mathbf{C}(\sigma) = \bigcap_{\tau<\sigma} \mathbf{C}(\tau)$, for $\sigma \in \mathbb{R}$. The direct inclusion is trivial. To show the converse inclusion, note that if $\sigma \leq 0$, there is nothing to show, whereas if $\sigma > 0$, property (*iii*) of Definition 6.3.19 implies that $\mathrm{Inv}(\xi_\sigma) \supseteq \bigcap_{\tau<\sigma} \mathrm{Inv}(\xi_\tau)$, so that $\mathbf{C}(\sigma) = \overline{\mathcal{S}} \cup (\mathcal{C} \cap \mathrm{Inv}(\xi_\sigma)) \supseteq \overline{\mathcal{S}} \cup (\mathcal{C} \cap \bigcap_{\tau<\sigma} \mathrm{Inv}(\xi_\tau)) = \bigcap_{\tau<\sigma}(\overline{\mathcal{S}} \cup (\mathcal{C} \cap \mathrm{Inv}(\xi_\tau))) = \bigcap_{\tau<\sigma} \mathbf{C}(\tau)$, by the infinite $\wedge$-distributivity of $\mathcal{P}(\mathcal{L})$. Finally, (6.46) follows easily from (6.45) and (6.11). Q.E.D.

The previous multiscale connectivity is referred to as a *contraction-pyramid multiscale connectivity*. In this framework, if $A \notin \mathcal{C}$, then A is fully disconnected, whereas if $A \in \mathcal{C}$, then its degree of connectivity is positive (nonnegative). If $A = O$ or $A \in \mathcal{S}$, then A is of course fully connected. On the other hand, if $A \in \mathcal{C} \setminus (\{O\} \cup \mathcal{S})$, then A is σ-connected if A is σ-stable, for $\sigma \in \mathbb{R}_+$. Hence, connectivity at positive scales corresponds to how stable a connected element A is with respect to the contraction pyramid. In most cases, this measures the "strength" of connectivity of A, with respect to the base connectivity $\mathcal{C}$.

The main example of a contraction pyramid is given by a granulometry that satisfies a certain smoothness condition. This is given by the following proposition. Recall that ψ° denotes the characteristic opening associated with an operator ψ.

6.3.21 Proposition. Let $\mathcal{L}$ be a lattice. A granulometry $\{\theta_\sigma \mid \sigma \in \mathbb{R}_+^*\}$ on $\mathcal{L}$, such that

$$\theta_\sigma = \left(\bigwedge_{\tau<\sigma} \theta_\tau \right)^\circ, \quad \sigma \in \mathbb{R}_+^*, \tag{6.47}$$

is a contraction pyramid on $\mathcal{L}$. □

Proof. Properties (*i*) and (*ii*) of Definition 6.3.19 are obvious. Now, for a given $\sigma \in \mathbb{R}_+^*$, note that $\bigwedge_{\tau<\sigma} \theta_\tau$ is an increasing and anti-extensive operator. We can then use Corollary 2.2.7 to conclude that $\mathrm{Inv}(\theta_\sigma) = \mathrm{Inv}\left((\bigwedge_{\tau<\sigma} \theta_\tau)^\circ\right) = \mathrm{Inv}\left(\bigwedge_{\tau<\sigma} \theta_\tau\right) = \bigcap_{\tau<\sigma} \mathrm{Inv}(\theta_\tau)$, which clearly implies property (*iii*) of Definition 6.3.19. Q.E.D.

The smoothness condition (6.47) implies that, for each $\sigma \in \mathbb{R}_+^*$, θ_σ is the greatest opening smaller than $\bigwedge_{\tau<\sigma} \theta_\tau$. A contraction pyramid $\{\theta_\sigma \mid \sigma \in \mathbb{R}_+^*\}$ that consists of openings is referred to as an *opening pyramid*, with associated *opening-pyramid multiscale connectivity* system $(\varphi, \mathbf{C})$, given by (6.45) and (6.46), with $\xi_\sigma = \theta_\sigma$, for $\sigma \in \mathbb{R}_+^*$. Moreover, in this case we say that A is σ-*open*, rather than σ-stable, if $\theta_\sigma(A) = A$, for $\sigma \in \mathbb{R}_+^*$ and $A \in \mathcal{L}$. Note that Theorem 6.1.13 implies that, for each $x \in \mathcal{S}$, the family $\{\gamma_{\sigma,x} \mid \sigma \in \mathbb{R}_+^*\}$

of σ-connectivity openings is a granulometry on $\mathcal{L}$ that satisfies (6.47); i.e., it is an opening pyramid.

Another example of an opening pyramid is given by the following proposition.

6.3.22 Proposition. Let E be a closed and bounded subset of $\mathbb{R}^n$, furnished with the Euclidean topology. Let $\mathcal{L} = \mathcal{F}(E)$, and let $B \in \mathcal{F}(E)$ be a structuring element.

(a) The restriction of the opening $\widehat{\theta}_\sigma(A) = A \circ \sigma B$ on $\mathcal{P}(E)$ to $\mathcal{F}(E)$ defines an opening θ_σ on $\mathcal{F}(E)$; i.e.,

$$A \in \mathcal{F}(E) \Rightarrow A \circ \sigma B \in \mathcal{F}(E), \quad \sigma \in \mathbb{R}_+^*. \tag{6.48}$$

In particular, $\{\theta_\sigma(A) = A \circ \sigma B \mid \sigma \in \mathbb{R}_+^*\}$ is a granulometry on $\mathcal{F}(E)$.

(b) If B is a Euclidean disk, then $\theta_\sigma = \bigwedge_{\tau<\sigma} \theta_\tau$, for $\sigma \in \mathbb{R}_+^*$; i.e., for $A \in \mathcal{F}(E)$,

$$A \circ \sigma B = \bigcap_{\tau<\sigma} A \circ \tau B, \quad \sigma \in \mathbb{R}_+^*. \tag{6.49}$$

In particular, the granulometry $\{\theta_\sigma(A) = A \circ \sigma B \mid \sigma \in \mathbb{R}_+^*\}$ satisfies condition (6.47) and is therefore an opening pyramid on $\mathcal{F}(E)$. □

Proof. (a): Since B is closed and E is bounded, B is compact in $\mathbb{R}^n$, and so is σB, for $\sigma \in \mathbb{R}_+$. Since A is closed in $\mathbb{R}^n$, it follows from [34, Lemma 7.42] that $A \circ \sigma B$ is closed in $\mathbb{R}^n$, so that $A \circ \sigma B \in \mathcal{F}(E)$, for $\sigma \in \mathbb{R}_+^*$.

(b): The inequality $A \circ \sigma B \subseteq \bigcap_{\tau<\sigma} A \circ \tau B$, for $\sigma \in \mathbb{R}_+^*$, is obvious. We show the reverse inequality. Without loss of generality, we assume that B is a disk of unit radius. Let $\sigma \in \mathbb{R}_+^*$ and $A \in \mathcal{F}(E)$. Consider the function $d_{\partial A}\colon A \to \mathbb{R}$, given by

$$d_{\partial A}(v) = \bigwedge \{d(v,w) \mid w \in \partial A\}, \quad v \in A, \tag{6.50}$$

where $d(u,v)$ denotes the Euclidean distance between u and v, and ∂A denotes the boundary of A. The value $d_{\partial A}(v)$ gives the distance of a point $v \in A$ to the boundary of A. Note that $d_{\partial A}$ is an infimum of continuous functions $d(\cdot, v)$, for $v \in d_{\partial A}$; hence, $d_{\partial A}$ is a u.s.c. function. From the definition of a structural opening, it is clear that

$$u \in A \circ \sigma B \;\Leftrightarrow\; \exists\, v \in A \text{ s.t. } d(u,v) \leq \sigma \leq d_{\partial A}(v). \tag{6.51}$$

Hence, if $u \in \bigcap_{\tau<\sigma} A \circ \tau B$, there is a sequence $\{v_{\tau_i}\} \subseteq A$ such that $\tau_i \uparrow \sigma$ and $d(u, v_{\tau_i}) \leq \tau_i \leq d_{\partial A}(v_{\tau_i})$, for each τ_i. Since A is closed and bounded, it is compact in the Euclidean topology. Hence, the sequence $\{v_{\tau_i}\}$ has a convergent subsequence $\{v_{\tau_{i_k}}\}$ that converges to a point $v \in A$. It then follows, by lower semi-continuity of $d(u, \cdot)$ on the left side and by upper semi-continuity of $d_{\partial A}$ on the right side, that $d(u,v) \leq \sigma \leq d_{\partial A}(v)$; i.e., $u \in A \bigcirc \sigma B$, as required. Q.E.D.

The following definition introduces the multiscale analog of locally-invariant openings (see Definition 4.3.17).

6.3.23 Definition. An opening pyramid $\{\theta_\sigma \mid \sigma \in \mathbb{R}^*_+\}$ on a lattice $\mathcal{L}$ is said to be *locally invariant* with respect to a connectivity class $\mathcal{C}$ in $\mathcal{L}$ if each opening θ_σ is locally invariant with respect to $\mathcal{C}$; i.e., if, for each $A \in \mathcal{L}$,

$$\theta_\sigma(A) = A \;\Rightarrow\; \theta_\sigma\gamma_x(A) = \gamma_x(A), \quad \sigma \in \mathbb{R}^*_+,\; x \in \mathcal{S}. \tag{6.52}$$

△

In other words, an opening pyramid on $\mathcal{L}$ is locally invariant if, given any $A \in \mathcal{L}$ such that A is σ-open, then each connected component of A is also σ-open, for $\sigma \in \mathbb{R}^*_+$.

We have the following result.

6.3.24 Lemma. Suppose that E is a Hausdorff space and $\widehat{\mathcal{C}}$ is a compatible connectivity class in $\mathcal{P}(E)$, with the points as sup-generators. Let $\widehat{\theta}$ be an opening on $\mathcal{P}(E)$, such that its restriction to $\mathcal{F}(E)$ defines an opening θ on $\mathcal{F}(E)$. If $\widehat{\theta}$ is locally invariant with respect to $\widehat{\mathcal{C}}$, then θ is locally invariant with respect to the connectivity class $\mathcal{C} = \widehat{\mathcal{C}} \cap \mathcal{F}(E)$. □

Proof. From Proposition 4.1.11, it follows that the connectivity openings $\{\gamma_x \mid x \in \mathcal{S}\}$ associated with $\mathcal{C}$ are the restriction to $\mathcal{F}(E)$ of the connectivity openings $\{\widehat{\gamma}_x \mid x \in \mathcal{S}\}$ associated with $\widehat{\mathcal{C}}$. Therefore, for all $A \in \mathcal{F}(E)$, we have that $\theta(A) = \widehat{\theta}(A) = A \Rightarrow \theta\gamma_x(A) = \widehat{\theta}\,\widehat{\gamma}_x(A) = \gamma_x(A) = \widehat{\gamma}_x(A)$, for $x \in \mathcal{S}$; i.e., $\theta(A) = A \Rightarrow \theta\gamma_x(A) = \gamma_x(A)$, for $x \in \mathcal{S}$, as required. Q.E.D.

By using the previous lemma and Corollary 4.3.19, we get the following result.

6.3.25 Proposition. Let E be a bounded and closed subset of $\mathbb{R}^n$, with the Euclidean topology. Let $\mathcal{L} = \mathcal{F}(E)$, with the points as sup-generators, furnished with the connectivity class $\mathcal{C}$ of topologically connected closed sets in E. For a connected structuring element $B \in \mathcal{F}(E)$, the opening $\theta_\sigma(A) = A \bigcirc \sigma B$ on $\mathcal{F}(E)$ is locally invariant with respect to $\mathcal{C}$, for $\sigma \in \mathbb{R}^*_+$. □

Proof. From Proposition 4.3.20(a), we have that θ_σ is the restriction of the operator $\widehat{\theta}_\sigma(A) = A \circ \sigma B$ on $\mathcal{P}(E)$ to $\mathcal{F}(E)$, for $\sigma \in \mathbb{R}_+^*$. Moreover, it follows easily from Corollary 4.3.19 that the granulometry $\{\widehat{\theta}_\sigma(A) \mid \sigma \in \mathbb{R}_+^*\}$ on $\mathcal{P}(E)$ is locally invariant with respect to the connectivity class $\widehat{\mathcal{C}}$ of the topologically connected sets in E. Note that $\mathcal{C} = \widehat{\mathcal{C}} \cap \mathcal{F}(E)$. It then follows from Lemma 6.3.24 that θ_σ is locally invariant with respect to $\mathcal{C}$. Q.E.D.

As a straightforward consequence of Proposition 4.3.20, it is possible to characterize the σ-connectivity openings and the σ-reconstruction operators associated with a multiscale connectivity generated by a locally-invariant opening pyramid. We have the following proposition.

6.3.26 Proposition. Let $\mathcal{L}$ be a lattice with a sup-generating family $\mathcal{S}$, furnished with a connectivity class $\mathcal{C}$. Let $(\varphi, \mathbf{C})$ be the multiscale connectivity system generated by a locally-invariant opening pyramid $\{\theta_\sigma \mid \sigma \in \mathbb{R}_+^*\}$ on $\mathcal{L}$, given by (6.45) and (6.46), with $\xi_\sigma = \theta_\sigma$, for $\sigma \in \mathbb{R}_+^*$. Let $\{\gamma_x \mid x \in \mathcal{S}\}$ and ρ be the connectivity openings and the reconstruction operator, respectively, associated with $\mathcal{C}$. Then:

(a) The σ-connectivity openings associated with $(\varphi, \mathbf{C})$ are given by

$$\gamma_{\sigma,x}(A) = \begin{cases} \gamma_x \theta_\sigma(A), & \text{if} \ \ x \leq \theta_\sigma(A) \\ x, & \text{if} \ \ \theta_\sigma(A) \not\geq x \leq A\,, \quad \sigma > 0, \\ O, & \text{if} \ \ x \not\leq A \end{cases} \tag{6.53}$$

and

$$\gamma_{\sigma,x}(A) = \gamma_x(A), \quad \sigma \leq 0, \tag{6.54}$$

for $A \in \mathcal{L}$, $x \in \mathcal{S}$.

(b) If $\mathcal{L}$ is infinite $\vee$-distributive, the σ-reconstruction operators associated with $(\varphi, \mathbf{C})$ are given by

$$\rho_\sigma(A \mid M) = \begin{cases} (A \wedge M) \vee \rho(\theta_\sigma(A) \mid M), & \text{for} \ \ \sigma > 0 \\ \rho(A \mid M), & \text{for} \ \ \sigma \leq 0 \end{cases}, \tag{6.55}$$

for $A, M \in \mathcal{L}$. □

The following corollary is an immediate consequence of part (a) of the previous proposition.

6.3.27 Corollary. Let $\mathcal{L}$ be a lattice with a sup-generating family $\mathcal{S}$, furnished with a connectivity class $\mathcal{C}$, with $\downarrow$-continuous connectivity openings. Let $(\varphi, \mathbf{C})$ be the multiscale connectivity system generated by a locally-invariant opening pyramid $\{\theta_\sigma \mid \sigma \in \mathbb{R}_+^*\}$ on $\mathcal{L}$, given by (6.45) and (6.46), with $\xi_\sigma = \theta_\sigma$, for $\sigma \in \mathbb{R}_+^*$. If, for each $\sigma \in \mathbb{R}_+^*$, the opening θ_σ is $\downarrow$-continuous, then the σ-connectivity openings $\{\gamma_{\sigma,x} \mid \sigma \in \mathbb{R}, x \in \mathcal{S}\}$ associated with $(\varphi, \mathbf{C})$ are $\downarrow$-continuous. □

The following result follows easily from Proposition 7.39 and Corollary 7.44, in [34].

6.3.28 Proposition. Let E be a bounded and closed subset of $\mathbb{R}^n$, furnished with the Euclidean topology. For $B \in \mathcal{F}(E)$, the operator $\theta_\sigma(A) = A \circ \sigma B$ on $\mathcal{F}(E)$ is $\downarrow$-continuous, for $\sigma \in \mathbb{R}_+$. □

Propositions 6.3.20, 6.3.22–6.3.28, and Corollary 6.3.27 lead to the following example.

6.3.29 Example. Let E be a connected, bounded and closed subset of $\mathbb{R}^n$, with the Euclidean topology. Let $\mathcal{L} = \mathcal{F}(E)$, with the points as sup-generators, furnished with the connectivity class $\mathcal{C}$ of topologically connected closed sets in E, and let $B \in \mathcal{F}(E)$ be a Euclidean disk. The granulometry $\{\theta_\sigma(A) = A \circ \sigma B \mid \sigma \in \mathbb{R}_+^*\}$ is a locally-invariant opening pyramid on $\mathcal{F}(E)$, with associated opening-pyramid multiscale connectivity system $(\varphi, \mathbf{C})$, given by (6.45) and (6.46), with $\xi_\sigma = \theta_\sigma$, for $\sigma \in \mathbb{R}_+^*$. Furthermore, the σ-connectivity openings $\{\gamma_{\sigma,x} \mid \sigma \in \mathbb{R}, x \in \mathcal{S}\}$ associated with $(\varphi, \mathbf{C})$ are $\downarrow$-continuous. ◊

Note that, in this framework, we have that $\varphi(A) = -\infty$, if $A \notin \mathcal{C}$, whereas $\varphi(A) = \bigvee\{\sigma \in \mathbb{R}_+^* \mid A \circ \sigma B = A\}$, if $A \in \mathcal{C} \setminus (\{O\} \cup \mathcal{S})$; i.e., the "supremum size" at which A is still σ-open. Fig. 6.5 provides an illustration, where E is a square subset of $\mathbb{R}^2$ with the Euclidean topology, and $B \in \mathcal{F}(E)$ is an Euclidean disk. Note that the original set A is connected, so that it has a positive degree of connectivity. Opening A by increasingly larger scaled replicas of the structuring element B eventually produces a set that differs from the original set. In this case, the (positive) degree of connectivity of A measures how strongly connected A is, with respect to the base connectivity $\mathcal{C}$.

The discretization of the theory of contraction-pyramid multiscale connectivity presented above is straightforward. The following is the discrete analog of Definition 6.3.19.

6.3.30 Definition. Let $\mathcal{L}$ be a lattice. A decreasing family $\{\xi_\sigma \mid \sigma \in \mathbb{Z}_+^*\}$ of contractions on $\mathcal{L}$ is said to be a *discrete contraction pyramid* on $\mathcal{L}$. △

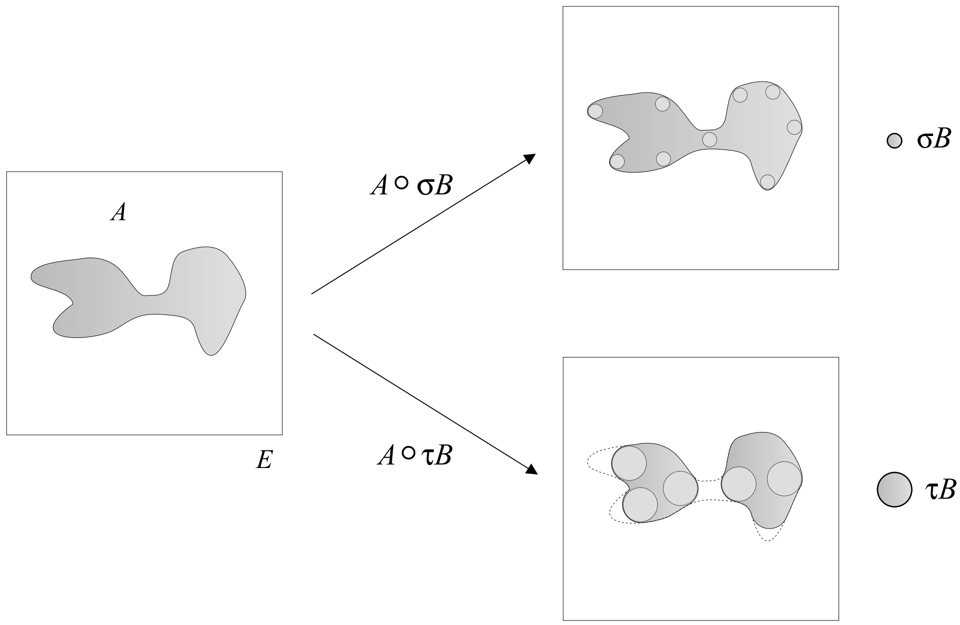


Figure 6.5: An example of opening-pyramid multiscale connectivity. The original set A is connected; hence, its degree of connectivity $\varphi(A)$ is positive. Note that A is σ-open, but not τ-open. Therefore, A is σ-connected, but not τ-connected. Equivalently, $\sigma \leq \varphi(A) < \tau$.

The following is the discrete analog of Proposition 6.3.20.

6.3.31 Proposition. Let $\mathcal{L}$ be an atomic lattice with sup-generating family $\mathcal{S}$, furnished with a connectivity class $\mathcal{C}$. If $\{\xi_\sigma \mid \sigma \in \mathbb{Z}_+^*\}$ is a discrete contraction pyramid on $\mathcal{L}$, then $\mathbf{C}$: $\mathbb{Z} \to \mathcal{P}(\mathcal{L})$, given by

$$\mathbf{C}(\sigma) = \begin{cases} \{O\} \cup \mathcal{S} \cup \{A \in \mathcal{C} \mid \xi_\sigma(A) = A\}, & \text{if } \sigma > 0 \\ \mathcal{C}, & \text{if } \sigma \leq 0 \end{cases}, \quad \sigma \in \mathbb{Z}, \tag{6.56}$$

is a discrete connectivity pyramid on $\mathcal{L}$, with associated discrete connectivity measure φ: $\mathcal{L} \to \overline{\mathbb{Z}}$ given by

$$\varphi(A) = \begin{cases} \infty, & \text{if } A = O \text{ or } A \in \mathcal{S} \\ \bigvee\{\sigma \in \mathbb{Z}_+^* \mid \xi_\sigma(A) = A\}, & \text{if } A \in \mathcal{C} \setminus (\{O\} \cup \mathcal{S}) \\ -\infty, & \text{if } A \notin \mathcal{C} \end{cases}, \quad A \in \mathcal{L}. \tag{6.57}$$

□

The discrete multiscale connectivity defined above is referred to as a *discrete contraction-pyramid multiscale connectivity*. All the remarks made previously regarding contraction-pyramid multiscale pyramids apply to the discrete case as well, with obvious modifications. The following result is easy to prove.

6.3.32 Proposition. Let $\mathcal{L}$ be a lattice. A granulometry $\{\theta_\sigma \mid \sigma \in \mathbb{Z}_+^*\}$ on $\mathcal{L}$ is a discrete contraction pyramid on $\mathcal{L}$. □

A discrete contraction pyramid $\{\theta_\sigma \mid \sigma \in \mathbb{Z}_+^*\}$ that consists of openings is referred to as a *discrete opening pyramid*, with associated *discrete opening-pyramid multiscale connectivity* system $(\varphi, \mathbf{C})$, given by (6.56) and (6.57), with $\xi_\sigma = \theta_\sigma$, for $\sigma \in \mathbb{Z}_+^*$. The definition of locally-invariant discrete opening pyramids and the discrete analog of Proposition 6.3.22 are straightforward and will not be repeated here.

The following example is a straightforward consequence of Corollary 4.3.19 and Propositions 6.3.31 and 6.3.32.

6.3.33 Example. Let E be a (not necessarily proper) subset of $\mathbb{R}^n$ or $\mathbb{Z}^n$, and let $\mathcal{L} = \mathcal{P}(E)$, with the points as sup-generators, furnished with a translation-invariant connectivity class $\mathcal{C}$. If $B \in \mathcal{C}$, then the granulometry $\{\theta_\sigma(A) = A \circ \sigma B \mid \sigma \in \mathbb{Z}_+^*\}$ is a locally-invariant discrete opening pyramid on $\mathcal{P}(E)$. The associated discrete opening-pyramid multiscale connectivity system $(\varphi, \mathbf{C})$ is given by (6.56) and (6.57), with $\xi_\sigma = \theta_\sigma$, for $\sigma \in \mathbb{Z}_+^*$. ◊

Note that, in this framework, we have that $\varphi(A) = -\infty$, if $A \notin \mathcal{C}$, whereas $\varphi(A) = m$, if $A \in \mathcal{C} \setminus (\{O\} \cup \mathcal{S})$, where m is the maximum size of the structuring element for which the object is still invariant under the structural opening. This provides a practical algorithm for computing $\varphi(A)$. Fig. 6.6 provides an illustration of this example, where E is a square subset of $\mathbb{Z}^2$, the base connectivity is given by 4-adjacency connectivity and the basic structuring element is a horizontal line of length 2. This figure depicts a 256-graylevel microscopic image of tracks in an electronic circuit and a binary version of this image, obtained by thresholding at level 200. Note that some tracks are faulty. Three tracks are considered separately, which leads to three discrete binary images A_1, A_2, and A_3, depicted in Fig. 6.6; their associated degrees of connectivity are also displayed. Tracks 2 and 6 have positive degree of connectivity, whereas the degree of connectivity of track 8 is $-\infty$, since this track is fully disconnected. Track 6 has a higher degree of connectivity than track 2, which indicates the track 6 is "more connected" than track 2.

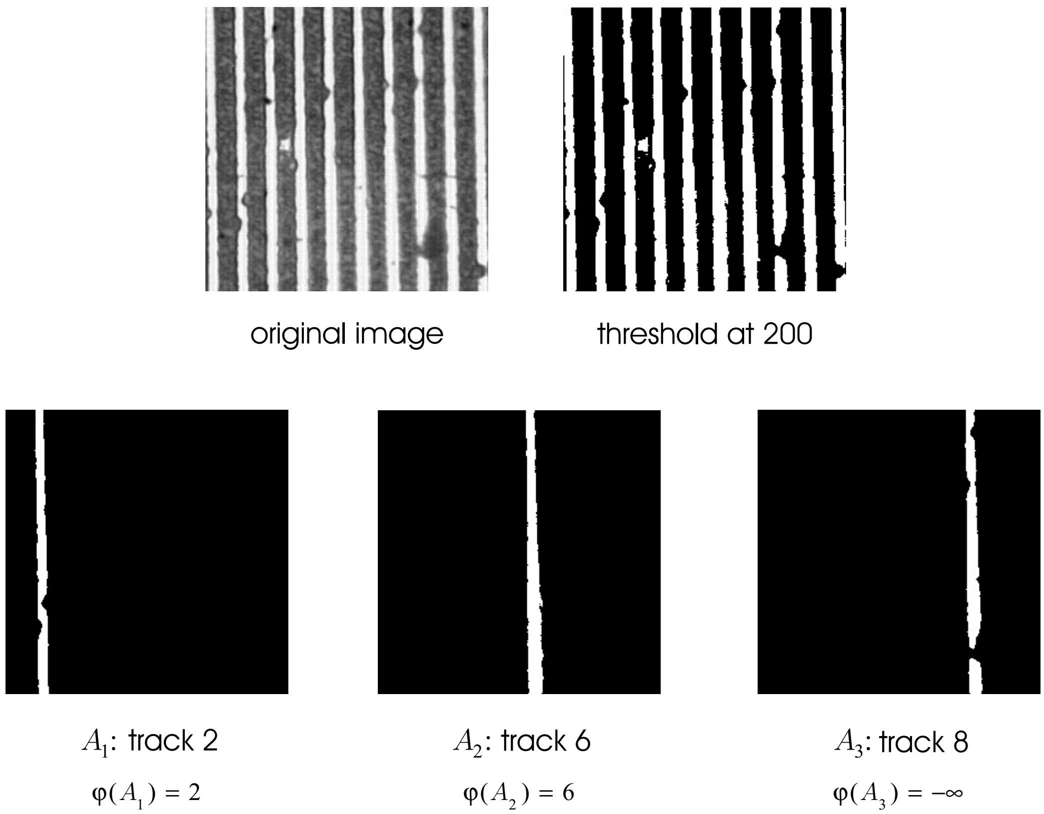


Figure 6.6: An example of discrete opening-pyramid multiscale connectivity. A horizontal line is used as structuring element. The original 256-graylevel microscopic image of tracks in an electronic circuit is binarized by thresholding at level 200. Three tracks are considered separately, leading to the discrete binary images A_1, A_2, and A_3. The more positive the degree of connectivity is, the stronger the connectivity of the particular object is.

6.4 Second-Generation Multiscale Connectivity

In this section, we study the problem of creating new multiscale connectivities from existing ones. Following Serra's nomenclature for the single-scale case, we refer to these new multiscale connectivities as *second-generation multiscale connectivities*. As it will become clear, this section extends many of the concepts and results discussed in Section 4.3 to the multiscale case.

6.4.1 Multiscale Connectivity Based on Clustering

Similarly to the single-scale case, we show next that it is possible to start with a given multiscale connectivity and obtain a second-generation multiscale connectivity based on clustering.

First, we study a class of operators that increase connectivity with respect to a given multiscale connectivity. We show that these operators are the multiscale analogs of connectivity-preserving operators in the single-scale case.

6.4.1 Definition. Let $\mathcal{L}$ be a lattice, furnished with a multiscale connectivity system $(\varphi, \mathbf{C})$. An operator ψ on $\mathcal{L}$ is said to be *connectivity-increasing* if

$$\varphi(\psi(A)) \geq \varphi(A), \quad \text{for all } A \in \mathcal{L}. \tag{6.58}$$

△

Therefore, a connectivity-increasing operator ψ can never decrease the degree of connectivity of an object. As might be expected, σ-connectivity openings are connectivity-increasing. This is shown by the following result.

6.4.2 Proposition. Let $\mathcal{L}$ be a lattice with sup-generating family $\mathcal{S}$, furnished with a multiscale connectivity system $(\varphi, \mathbf{C})$. The σ-connectivity openings $\{\gamma_{\sigma,x} \mid \sigma \in \mathbb{R}, x \in \mathcal{S}\}$ associated with $(\varphi, \mathbf{C})$ are connectivity-increasing. □

Proof. Let $A \in \mathcal{L}$ and $\sigma \in \mathbb{R}$. If $x \not\leq A$, then $\gamma_{\sigma,x}(A) = O$, which trivially implies that $\varphi(\gamma_{\sigma,x}(A)) \geq \varphi(A)$. Hence, we can assume that $x \leq A$. If $\varphi(A) \geq \sigma$, we have that $A \in \mathbf{C}(\sigma) \Rightarrow \gamma_{\sigma,x}(A) = A$; hence, $\varphi(\gamma_{\sigma,x}(A)) = \varphi(A)$. On the other hand, if $\varphi(A) < \sigma$, then $\gamma_{\sigma,x}(A) \in \mathbf{C}(\sigma) \Rightarrow \varphi(\gamma_{\sigma,x}(A)) \geq \sigma > \varphi(A)$. In any case, we get $\varphi(\gamma_{\sigma,x}(A)) \geq \varphi(A)$, as required. Q.E.D.

The next result shows that connectivity-increasing operators constitute the multiscale extension of connectivity-preserving operators.

6.4.3 Proposition. Let $\mathcal{L}$ be a lattice, and let $(\varphi, \mathbf{C})$ be a multiscale connectivity system on $\mathcal{L}$. An operator ψ on $\mathcal{L}$ is connectivity-increasing if and only if it is connectivity-preserving with respect to each level of $\mathbf{C}$; i.e., $\psi(\mathbf{C}(\sigma)) \subseteq \mathbf{C}(\sigma)$, for each $\sigma \in \mathbb{R}$. □

Proof. "$\Rightarrow$": For a given $\sigma \in \mathbb{R}$ and $C \in \mathbf{C}(\sigma)$, we have that $\varphi(\psi(C)) \geq \varphi(C) \geq \sigma \Rightarrow \psi(C) \in \mathbf{C}(\sigma)$, so that $\psi(\mathbf{C}(\sigma)) \subseteq \mathbf{C}(\sigma)$, as required.

"$\Leftarrow$": Let $A \in \mathcal{L}$, and $\varphi(A) = \sigma_0$. If $\sigma_0 = -\infty$, then $\varphi(\psi(A)) \geq \varphi(A) = -\infty$. If $\sigma_0 \in \mathbb{R}$, it follows that $A \in \mathbf{C}(\sigma_0) \Rightarrow \psi(A) \in \mathbf{C}(\sigma_0) \Rightarrow \varphi(\psi(A)) \geq \sigma_0 = \varphi(A)$. Finally, if $\sigma_0 = \infty$, it follows that $A \in \mathbf{C}(\sigma) \Rightarrow \psi(A) \in \mathbf{C}(\sigma)$, for all $\sigma \in \mathbb{R}$, so that $\varphi(\psi(A)) = \varphi(A) = \infty$. Combining these three cases, we get $\varphi(\psi(A)) \geq \varphi(A)$, as required. Q.E.D.

Note that the previous result leads to an alternative proof of Proposition 6.4.2. In addition, by using the fact that an extensive dilation δ on $\mathcal{L}$ is connectivity-preserving if and only if $\delta(x) \in \mathcal{C}$, for all $x \in \mathcal{S}$ (see the proof of Proposition 4.3.9(a)), we can easily show the following result.

6.4.4 Proposition. Let $\mathcal{L}$ be a lattice with a sup-generating family $\mathcal{S}$, furnished with a multiscale connectivity system $(\varphi, \mathbf{C})$. An extensive dilation δ on $\mathcal{L}$ is connectivity-increasing if and only if $\delta(x)$ is fully connected, for every $x \in \mathcal{S}$. □

For instance, let $\mathcal{L} = \mathcal{P}(E)$, where $E = \mathbb{R}^n$ or $E = \mathbb{Z}^n$, with the points as sup-generators, and let $(\varphi, \mathbf{C})$ be a translation-invariant multiscale connectivity system on $\mathcal{P}(E)$. If B is a fully connected structuring element that contains the origin of E, then clearly the translation-invariant dilation $\delta_B(A) = A \oplus B$ is a connectivity-increasing operator on $\mathcal{P}(E)$. For a particular case, consider $\mathcal{L} = \mathcal{P}(\mathbb{Z}^2)$, with the points as sup-generators, and let $(\varphi, \mathbf{C})$ be the translation-invariant multiscale connectivity system associated with Example 6.1.2. If B is the cross structuring element, then the dilation $\delta_B(A) = A \oplus B$ will always increase connectivity on $\mathcal{P}(\mathbb{Z}^2)$.

In Section 4.3.1, we defined clusterings. Next, we define multiscale clusterings.

6.4.5 Definition. Let $\mathcal{L}$ be a lattice, furnished with a multiscale connectivity system $(\varphi, \mathbf{C})$. An operator ψ on $\mathcal{L}$ is said to be a *multiscale clustering* if ψ is a clustering with respect to each level of $\mathbf{C}$. A multiscale clustering ψ is said to be *strong* if ψ is a strong clustering with respect to each level of $\mathbf{C}$; i.e., if

$$\psi(\mathbf{id} \wedge \gamma_{\sigma,x}\psi) = \gamma_{\sigma,x}\psi, \quad \sigma \in \mathbb{R},\ x \in \mathcal{S}, \tag{6.59}$$

where $\{\gamma_{\sigma,x} \mid \sigma \in \mathbb{R}, x \in \mathcal{S}\}$ are the σ-connectivity openings associated with $(\varphi, \mathbf{C})$. △

The following result provides a characterization of multiscale clusterings.

6.4.6 Proposition. Let $\mathcal{L}$ be a lattice with a sup-generating family $\mathcal{S}$, furnished with a multiscale connectivity system $(\varphi, \mathbf{C})$. An operator ψ on $\mathcal{L}$ is a multiscale clustering if and only if:

(i) ψ is increasing and extensive.

(ii) ψ is connectivity-increasing.

(iii) For a family $\{A_\alpha\}$ in $\mathcal{L}$ such that $\bigwedge A_\alpha \neq O$, we have $\varphi(\psi(\bigvee A_\alpha)) \geq \bigwedge \varphi(\psi(A_\alpha))$. □

Proof. It follows from Definitions 4.3.4 and 6.4.5 that ψ is a multiscale clustering if and only if: (i') it is increasing and extensive, (ii') it is connectivity-preserving at each scale, and (iii') for a family $\{A_\alpha\}$ in $\mathcal{L}$ such that $\bigwedge A_\alpha \neq O$, we have that $\psi(A_\alpha) \in \mathbf{C}(\sigma)$, for all α $\Rightarrow \psi(\bigvee A_\alpha) \in \mathbf{C}(\sigma)$, for each $\sigma \in \mathbb{R}$. Conditions (i) and (i') are the same, while conditions (ii) and (ii') are equivalent, by Proposition 6.4.3. It remains to show that conditions (iii) and (iii') are equivalent. Let $\{A_\alpha\}$ be a family in $\mathcal{L}$ such that $\bigwedge A_\alpha \neq O$. We show that $(iii') \Rightarrow (iii)$. Let $\sigma_0 = \bigwedge \varphi(\psi(A_\alpha))$. If $\sigma_0 = -\infty$, there is nothing to prove. If $\sigma_0 \in \mathbb{R}$, we have $\varphi(\psi(A_\alpha)) \geq \sigma_0 \Rightarrow \psi(A_\alpha) \in \mathbf{C}(\sigma_0)$, for all $\alpha \Rightarrow \psi(\bigvee A_\alpha) \in \mathbf{C}(\sigma_0) \Rightarrow \varphi(\psi(\bigvee A_\alpha)) \geq \sigma_0$. Finally, if $\sigma_0 = \infty$, it follows that, for each α, $\varphi(\psi(A_\alpha)) = \infty \Rightarrow A_\alpha \in \mathbf{C}(\sigma)$, for all $\sigma \in \mathbb{R}$ $\Rightarrow \psi(\bigvee A_\alpha) \in \mathbf{C}(\sigma)$, for all $\sigma \in \mathbb{R} \Rightarrow \varphi(\psi(\bigvee A_\alpha)) = \infty = \sigma_0$. Combining these three cases, we get $\varphi(\psi(\bigvee A_\alpha)) \geq \sigma_0 = \bigwedge \varphi(\psi(A_\alpha))$, as required. To show that $(iii) \Rightarrow (iii')$, note that, given $\sigma \in \mathbb{R}$, we have $\psi(A_\alpha) \in \mathbf{C}(\sigma)$, for all $\alpha \Rightarrow \bigwedge \varphi(\psi(A_\alpha)) \geq \sigma$, so that $\varphi(\psi(\bigvee A_\alpha)) \geq \bigwedge \varphi(\psi(A_\alpha)) \geq \sigma \Rightarrow \psi(\bigvee A_\alpha) \in \mathbf{C}(\sigma)$, as required. Q.E.D.

As an example, it follows from the above result and Proposition 6.4.4 that an extensive dilation δ on $\mathcal{L}$ such that $\delta(x)$ is fully connected, for every $x \in \mathcal{S}$, is a multiscale clustering on $\mathcal{L}$ (this also follows easily from Proposition 4.3.9(a)). Another example of multiscale clustering is given by connectivity-increasing closings, since, by Proposition 4.3.8, connectivity-preserving closings are clusterings.

The following result is the multiscale version of Proposition 4.3.5.

6.4.7 Proposition. Let $\mathcal{L}$ be a lattice with a sup-generating family $\mathcal{S}$, furnished with a multiscale connectivity system $(\varphi, \mathbf{C})$, and let ψ be a multiscale clustering on $\mathcal{L}$. We have that:

(a) $\varphi^\psi = \varphi(\psi(\cdot))$ is a connectivity measure on $\mathcal{L}$, such that $\varphi \leq \varphi^\psi$.

(b) $\mathbf{C}^\psi\colon \mathbb{R} \to \mathcal{P}(\mathcal{L})$, given by

$$\mathbf{C}^\psi(\sigma) = \psi^{-1}(\mathbf{C}(\sigma)) = \{A \in \mathcal{L} \mid \psi(A) \in \mathbf{C}(\sigma)\}, \quad \text{for } \sigma \in \mathbb{R}, \tag{6.60}$$

is a connectivity pyramid on $\mathcal{L}$, such that $\mathbf{C} \leq \mathbf{C}^\psi$.

(c) $(\varphi^\psi, \mathbf{C}^\psi)$ constitutes a multiscale connectivity system on $\mathcal{L}$.

(d) For $A \in \mathcal{L}$, the σ-partition of the HPCC $\mathbf{c}_A^\psi$ of A according to $(\varphi^\psi, \mathbf{C}^\psi)$ is coarser than the σ-partition of the HPCC $\mathbf{c}_A$ of A according to $(\varphi, \mathbf{C})$ (i.e., $\mathbf{c}_A^\psi$ is *coarser* than $\mathbf{c}_A$):

$$\mathbf{c}_A(\sigma, x) = \gamma_{\sigma,x}(A) \leq \gamma^\psi_{\sigma,x}(A) = \mathbf{c}_A^\psi(\sigma, x), \quad \text{for all } \sigma \in \mathbb{R},\ x \leq A, \tag{6.61}$$

where $\{\gamma_{\sigma,x} \mid \sigma \in \mathbb{R}, x \in \mathcal{S}\}$ and $\{\gamma^{\psi}_{\sigma,x} \mid \sigma \in \mathbb{R}, x \in \mathcal{S}\}$ are the σ-connectivity openings associated with $(\varphi, \mathbf{C})$ and $(\varphi^{\psi}, \mathbf{C}^{\psi})$, respectively. □

Proof. (a): Note that $\varphi^{\psi}(O) = \varphi(\psi(O)) \geq \varphi(O) = \infty \Rightarrow \varphi^{\psi}(O) = \infty$, since ψ is connectivity-increasing. By using the same argument, we can show that $\varphi^{\psi}(x) = \infty$, for $x \in \mathcal{S}$. This shows axiom (i) of a connectivity measure. Axiom (ii) follows directly from item (iii) of Proposition 6.4.6. The inequality $\varphi \leq \varphi^{\psi}$ is a direct consequence of the fact that ψ is connectivity-increasing: $\varphi^{\psi}(A) = \varphi(\psi(A)) \geq \varphi(A)$, for all $A \in \mathcal{L}$.

(b): Let Γ be the mapping from $\mathcal{M}(\mathcal{L})$ into $\mathcal{Y}(\mathcal{L})$, given by (6.10). From part (a), we have that $\varphi^{\psi} \in \mathcal{M}(\mathcal{L})$. Now, $\Gamma(\varphi^{\psi}(\sigma)) = \{A \in \mathcal{L} \mid \varphi^{\psi}(A) \geq \sigma\} = \{A \in \mathcal{L} \mid \varphi(\psi(A)) \geq \sigma\} = \{A \in \mathcal{L} \mid \psi(A) \in \mathbf{C}(\sigma)\} = \mathbf{C}^{\psi}(\sigma)$, for $\sigma \in \mathbb{R}$. Therefore, $\mathbf{C}^{\psi} \in \mathcal{Y}(\mathcal{L})$; i.e., $\mathbf{C}^{\psi}$ is a connectivity pyramid on $\mathcal{L}$. The inequality $\mathbf{C} \leq \mathbf{C}^{\psi}$ follows from $\varphi \leq \varphi^{\psi}$ and the fact that Γ is order-preserving.

(c): This is a consequence of the argument used in the proof of part (b).

(d): This follows easily from the fact that $\mathbf{C} \leq \mathbf{C}^{\psi}$. Q.E.D.

The multiscale connectivity system $(\varphi^{\psi}, \mathbf{C}^{\psi})$ of Proposition 6.4.7 is said to generate a *clustering-based second-generation multiscale connectivity* on $\mathcal{L}$. This multiscale connectivity is "richer" than the original multiscale connectivity, in the sense that every element of the lattice has a higher degree of connectivity, there are more σ-connected elements at each scale σ, and the σ-partitions of connected components are coarser.

Finally, we give the multiscale version of Proposition 4.3.6. The proof is omitted, since it is a direct extension of the proof of the single-scale result.

6.4.8 Proposition. Let $\mathcal{L}$ be a lattice with a sup-generating family $\mathcal{S}$, furnished with a multiscale connectivity system $(\varphi, \mathbf{C})$. Let $\{\gamma_{\sigma,x} \mid \sigma \in \mathbb{R}, x \in \mathcal{S}\}$ and $\{\rho_{\sigma} \mid \sigma \in \mathbb{R}\}$ be the σ-connectivity openings and the σ-reconstruction operators, respectively, associated with $(\varphi, \mathbf{C})$. If ψ is a strong multiscale clustering on $\mathcal{L}$, then:

(a) The σ-connectivity openings associated with $(\varphi^{\psi}, \mathbf{C}^{\psi})$ are given by

$$\gamma^{\psi}_{\sigma,x}(A) = \begin{cases} A \wedge \gamma_{\sigma,x}\psi(A), & \text{if } x \leq A \\ O, & \text{if } x \not\leq A \end{cases}, \quad \sigma \in \mathbb{R},\ x \in \mathcal{S}, \tag{6.62}$$

for $A \in \mathcal{L}$.

(b) If $\mathcal{L}$ is infinite $\vee$-distributive, the σ-reconstruction operators associated with $(\varphi^{\psi}, \mathbf{C}^{\psi})$ are given by

$$\rho_{\sigma}^{\psi}(A \mid M) = A \wedge \rho_{\sigma}(\psi(A) \mid A \wedge M), \quad \sigma \in \mathbb{R}, \tag{6.63}$$

for $A, M \in \mathcal{L}$. □

6.4.2 Multiscale Connectivity Based on Contraction

Another way to construct a new multiscale connectivity from an existing one is by means of a contraction.

The following is the multiscale version of Proposition 4.3.16.

6.4.9 Proposition. Consider an atomic lattice $\mathcal{L}$ with sup-generating family $\mathcal{S}$, furnished with a multiscale connectivity system $(\varphi, \mathbf{C})$. Let ξ be a contraction on $\mathcal{L}$. The mapping $\mathbf{C}^{\xi}\colon \mathbb{R} \to \mathcal{P}(\mathcal{L})$, given by

$$\mathbf{C}^{\xi}(\sigma) = \{O\} \cup \mathcal{S} \cup \{A \in \mathbf{C}(\sigma) \mid \xi(A) = A\}, \quad \sigma \in \mathbb{R}, \tag{6.64}$$

is a connectivity pyramid on $\mathcal{L}$, such that $\mathbf{C}^{\xi} \leq \mathbf{C}$, with associated connectivity measure $\varphi^{\xi}\colon \mathcal{L} \to \overline{\mathbb{R}}$, given by

$$\varphi^{\xi}(A) = \begin{cases} \varphi(A), & \text{if } A = O \text{ or } A \in \mathcal{S} \text{ or } \xi(A) = A \\ -\infty, & \text{otherwise} \end{cases}, \quad A \in \mathcal{L}, \tag{6.65}$$

such that $\varphi^{\xi} \leq \varphi$. □

PROOF. From Proposition 4.3.16, we have that $\mathbf{C}^{\xi}(\sigma)$ is a connectivity class in $\mathcal{L}$, for each $\sigma \in \mathbb{R}$, which shows axiom (i) of a connectivity pyramid. Clearly, $\mathbf{C}(\sigma) \subseteq \mathbf{C}(\tau)$ implies that $\mathbf{C}^{\xi}(\sigma) \subseteq \mathbf{C}^{\xi}(\tau)$, for $\sigma \geq \tau$, which shows axiom (ii) of a connectivity pyramid. To verify axiom (iii), note that $\mathbf{C}^{\xi}(\sigma) = \{O\} \cup \mathcal{S} \cup [\mathbf{C}(\sigma) \cap \mathrm{Inv}(\xi)]$. Therefore, $\mathbf{C}^{\xi}(\sigma) = \{O\} \cup \mathcal{S} \cup [(\bigcap_{\tau<\sigma} \mathbf{C}(\tau)) \cap \mathrm{Inv}(\xi)] = \{O\} \cup \mathcal{S} \cup \bigcap_{\tau<\sigma}[\mathbf{C}(\tau) \cap \mathrm{Inv}(\xi)] = \bigcap_{\tau<\sigma}(\{O\} \cup \mathcal{S} \cup [\mathbf{C}(\tau) \cap \mathrm{Inv}(\xi)]) = \bigcap_{\tau<\sigma} \mathbf{C}^{\xi}(\tau)$, since $\mathcal{P}(\mathcal{L})$ is infinite $\wedge$-distributive. Moreover, it is obvious that $\mathbf{C}^{\xi}(\sigma) \subseteq \mathbf{C}(\sigma)$, for each $\sigma \in \mathbb{R}$; i.e., $\mathbf{C}^{\xi} \leq \mathbf{C}$. Finally, (6.65) follows easily from (6.64) and (6.11), whereas the inequality $\varphi^{\xi} \leq \varphi$ is a direct consequence of (6.65) or, alternatively, of the fact that $\mathbf{C}^{\xi} \leq \mathbf{C}$. Q.E.D.

The multiscale connectivity system $(\varphi^\xi, \mathbf{C}^\xi)$ of Proposition 6.4.9 is said to generate a *contraction-based second-generation multiscale connectivity* on $\mathcal{L}$. Note that all elements of $\mathcal{L}$ that are not invariant to ξ become fully disconnected in the contraction-based multiscale connectivity (this affords robustness against noise). Therefore, the new multiscale connectivity is "stricter" than the original one; i.e., there are fewer σ-connected elements at each scale σ.

In the case in which ξ is an opening $\xi = \theta$ on $\mathcal{L}$, the multiscale connectivity system $(\varphi^\theta, \mathbf{C}^\theta)$, given by (6.64) and (6.65), with $\xi = \theta$, defines an *opening-based second-generation multiscale connectivity.*

As in the single-scale case, the following property of θ allows one to characterize the σ-connectivity openings and the σ-reconstruction operators associated with an opening-based multiscale connectivity.

6.4.10 Definition. An opening θ on a lattice $\mathcal{L}$ is said to be locally invariant with respect to a multiscale connectivity system $(\varphi, \mathbf{C})$ on $\mathcal{L}$ if

$$\theta(A) = A \;\Rightarrow\; \theta\gamma_{\sigma,x}(A) = \gamma_{\sigma,x}(A), \quad \sigma \in \mathbb{R}, x \in \mathcal{S}. \tag{6.66}$$

where $\{\gamma_{\sigma,x} \mid \sigma \in \mathbb{R}, x \in \mathcal{S}\}$ are the σ-connectivity openings associated with $(\varphi, \mathbf{C})$. $\triangle$

Note that the previous definition is the multiscale extension of Definition 4.3.17.

The following is the multiscale version of Proposition 4.3.18. The proof is omitted, since it is a straightforward extension of the proof of the single-scale result.

6.4.11 Proposition. Let $\mathcal{L}$ be an infinite $\vee$-distributive lattice with a sup-generating family $\mathcal{S}$, furnished with a multiscale connectivity system $(\varphi, \mathbf{C})$. An opening θ on $\mathcal{L}$ is locally invariant with respect to $(\varphi, \mathbf{C})$ if and only if there exists a family $\mathcal{B}$ of fully connected elements (i.e., $\mathcal{B} \subseteq \bigcap_{\sigma\in\mathbb{R}} \mathbf{C}(\sigma)$), such that

$$\theta(A) = \theta_{\mathcal{B}}(A) = \bigvee\{B \in \mathcal{B} \mid B \leq A\}. \tag{6.67}$$

$\square$

A straightforward corollary of the previous result is that, given a dilation δ on $\mathcal{L}$ such that $\delta(x)$ is fully connected, for each $x \in \mathcal{S}$, the adjunctional opening $\theta = \delta\epsilon$ on $\mathcal{L}$ is locally invariant with respect to $(\varphi, \mathbf{C})$ (see also Corollary 4.3.19). For instance, consider a translation-invariant multiscale connectivity system $(\varphi, \mathbf{C})$ on $\mathcal{L} = \mathcal{P}(\mathbb{R}^n)$. If B is a fully connected structuring element, then the structural opening $\theta_B(A) = A \circ B$ is locally invariant with respect to $(\varphi, \mathbf{C})$.

The usefulness of locally-invariant openings becomes clear from the following result, which is the multiscale version of Proposition 4.3.20. The proof is omitted, since it is a direct extension of the proof of the single-scale result.

6.4.12 Proposition. Consider an atomic lattice $\mathcal{L}$ with sup-generating family $\mathcal{S}$, furnished with a multiscale connectivity system $(\varphi, \mathbf{C})$. Let $\{\gamma_{\sigma,x} \mid \sigma \in \mathbb{R}, x \in \mathcal{S}\}$ and $\{\rho_\sigma \mid \sigma \in \mathbb{R}\}$ be the σ-connectivity openings and the σ-reconstruction operators, respectively, associated with $(\varphi, \mathbf{C})$. Let θ be an opening on $\mathcal{L}$ that is locally invariant with respect to $(\varphi, \mathbf{C})$, and let $(\varphi^\theta, \mathbf{C}^\theta)$ be the opening-based multiscale connectivity system generated by θ, given by (6.64) and (6.65), with $\xi = \theta$.

(a) The σ-connectivity openings associated with $(\varphi^\theta, \mathbf{C}^\theta)$ are given by

$$\gamma^\theta_{\sigma,x}(A) = \begin{cases} \gamma_{\sigma,x}\theta(A), & \text{if } \; x \leq \theta(A) \\ x, & \text{if } \; \theta(A) \not\geq x \leq A\,, \\ O, & \text{if } \; x \not\leq A \end{cases} \quad \sigma \in \mathbb{R},\ x \in \mathcal{S}, \tag{6.68}$$

for $A \in \mathcal{L}$.

(b) The σ-reconstruction operators associated with $(\varphi^\theta, \mathbf{C}^\theta)$ are given by

$$\rho^\theta_\sigma(A \mid M) = (A \wedge M) \vee \rho_\sigma(\theta(A) \mid M), \quad \sigma \in \mathbb{R}, \tag{6.69}$$

for $A, M \in \mathcal{L}$. □

6.5 Multiscale Tools

In this section, we discuss the application of multiscale connectivities to image analysis tasks, such as pyramid decompositions, hierarchical segmentation and hierarchical clustering, and multiscale features.

6.5.1 Pyramid Decompositions

Discrete multiscale connectivities lead to an interesting example of a nonlinear multiscale signal decomposition scheme. The basic idea is to use σ-reconstruction operators as the analysis operators of a nonlinear pyramid decomposition scheme. In contrast to pyramid decompositions based on pixel-based linear or nonlinear operators [27], the pyramid decomposition that we propose here does not work at the pixel level, but at the level of the connected components of an object at various scales. This leads to a novel *object-based* multiscale signal decomposition scheme. This scheme could be thought of as the pyramid transform analog of the so-called *second-generation* image coding techniques [40], which constitute an object-based approach to image compression that codes homogeneous regions (objects) in an image.

In the following, we briefly outline a few basic aspects of the theory of multiscale signal decomposition. For a more detailed treatment, the reader is referred to [27].

Consider a family $\{V_\sigma \mid \sigma \in \mathbb{Z}\}$ of *multiscale spaces*, such that the *nesting condition* $V_\sigma \subseteq V_\tau$, if $\sigma \geq \tau$, is satisfied (this condition is a basic requirement for a multiscale signal decomposition scheme, as argued in [53]). The space V_σ contains the signals of interest at scale σ. In addition, consider a family $\{\psi_\sigma^\uparrow : V_\sigma \to V_{\sigma+1} \mid \sigma \in \mathbb{Z}\}$ of *analysis operators* and a family $\{\psi_\sigma^\downarrow : V_{\sigma+1} \to V_\sigma \mid \sigma \in \mathbb{Z}\}$ of *synthesis operators*. The analysis operator $\psi_\sigma^\uparrow$ maps a signal at scale σ to scale $\sigma+1$, reducing information in the process, whereas the synthesis operator $\psi_\sigma^\downarrow$ maps this information back to scale σ. The composition $\psi_\sigma^\downarrow\psi_\sigma^\uparrow$ is an operator on V_σ, known as the *approximation operator*. On the other hand, the composition $\psi_\sigma^\uparrow\psi_\sigma^\downarrow$ is an operator on $V_{\sigma+1}$ that must always be the identity operator. The condition $\psi_\sigma^\uparrow\psi_\sigma^\downarrow = \mathbf{id}$ on $V_{\sigma+1}$ is referred to as the *pyramid condition* (in [27], it is shown that the pyramid condition is a basic requirement for a multiscale decomposition scheme).

Assume that, for each $\sigma \in \mathbb{Z}$, there is a *subtraction operation* $\dot{-} : V_\sigma \times V_\sigma \to Y_\sigma$, where Y_σ is a *difference space*, and an *addition operation* $\dot{+} : V_\sigma \times Y_\sigma \to V_\sigma$, such that

$$\psi_\sigma^\downarrow\psi_\sigma^\uparrow(A) \dot{+} (A \dot{-} \psi_\sigma^\downarrow\psi_\sigma^\uparrow(A)) \;=\; A, \quad A \in V_\sigma. \tag{6.70}$$

This equation is referred to as the *perfect reconstruction condition*. Let A be a signal in a given multiscale space V_{σ_0}, where $\sigma_0 \in \mathbb{Z}$. The *pyramid transform* $\{D_0, D_1, \ldots, D_{m-1}, A_m\}$ of A is given by the following recursion:

$$A \to \{D_0, A_1\} \to \{D_0, D_1, A_2\} \to \cdots \to \{D_0, D_1, \ldots, D_{m-1}, A_m\}, \tag{6.71}$$

for $m \geq 1$, with

$$\begin{cases} A_{j+1} \;=\; \psi_{\sigma_0+j}^\uparrow(A_j) \in V_{\sigma_0+j+1} \\ D_j \;=\; A_j \dot{-} \psi_{\sigma_0+j}^\downarrow(A_{j+1}) \in Y_{\sigma_0+j} \end{cases}, \tag{6.72}$$

for $j = 0, 1, \ldots, m-1$, where $A_0 = A \in V_{\sigma_0}$. The *basic signal* A_m is the coarsest scaled version of the original signal A, whereas each *detail signal* D_j expresses the information contained in A_j that is lost in the approximation $\psi^{\downarrow}_{\sigma_0+j}\psi^{\uparrow}_{\sigma_0+j}(A_j)$, for $j = 0, 1, \ldots, m-1$.

Due to the perfect reconstruction condition (6.70), the original signal A can be exactly reconstructed from $\{D_0, D_1, \ldots, D_{m-1}, A_m\}$, by means of the *inverse pyramid transform*, given by the backward recursion:

$$A_j = \psi^{\downarrow}_{\sigma_0+j}(A_{j+1}) \dot{+} D_j, \tag{6.73}$$

for $j = 0, 1, \ldots, m-1$, with $A = A_0$. The inverse pyramid transform propagates the information from scale $\sigma_0 + m$ back to scale σ_0, by adding, at each intermediate scale, the corresponding detail signal; i.e., the information that had been missing at that scale.

The multiscale signal decomposition scheme discussed above has obvious applications in image coding for image compression or progressive transmission. The basic signal corresponds to the information that needs to be preserved, or transmitted first, whereas the detail signals correspond to the information that can be either discarded, quantized, or transmitted at later successive times. Usually, the detail signals at lower scales (lower indices) correspond to information that can be severely quantized or transmitted last.

We now present an interesting example of multiscale signal decomposition scheme based on the σ-reconstruction operators of a discrete multiscale connectivity. Let $\mathcal{L}$ be a lattice sup-generated by $\mathcal{S}$, furnished with a discrete multiscale connectivity system $(\varphi, \mathbf{C})$. Let the multiscale spaces $\{V_\sigma \mid \sigma \in \mathbb{Z}\}$ be given by

$$V_\sigma = \mathrm{Inv}(\rho_\sigma(\cdot \mid R)), \quad \sigma \in \mathbb{Z}, \tag{6.74}$$

where $\{\rho_\sigma \mid \sigma \in \mathbb{Z}\}$ is the family of σ-reconstruction operators associated with $(\varphi, \mathbf{C})$. For reasons that will become clear later, the fixed marker $R \in \mathcal{L}$ is called the *root marker*. It is easy to see that the nesting condition $V_\sigma \subseteq V_\tau$, if $\sigma \geq \tau$, is satisfied, due to the fact that $\{\rho_\sigma(\cdot \mid M) \mid \sigma \in \mathbb{Z}\}$ constitutes a granulometry on $\mathcal{L}$ (see Theorem 6.2.12).

In the case of an infinite $\vee$-distributive lattice, we have the following characterization of the multiscale spaces $\{V_\sigma \mid \sigma \in \mathbb{Z}\}$.

6.5.1 Proposition. Let $\mathcal{L}$ be an infinite $\vee$-distributive lattice. For each $\sigma \in \mathbb{Z}$, we have that $A \in V_\sigma$ if and only if $C \wedge R \neq O$, for all $C \lessdot_\sigma A$. □

Proof. "$\Rightarrow$": Given $\sigma \in \mathbb{Z}$, suppose that there is $C_0 \lessdot_\sigma A$ such that $C_0 \wedge R = O$. It would follow, by the infinite $\vee$-distributivity of $\mathcal{L}$, that $C_0 = C_0 \wedge A = C_0 \wedge \rho_\sigma(A \mid R) = C_0 \wedge \bigvee\{C \mid C \lessdot_\sigma A, C \wedge R \neq O\} = \bigvee\{C_0 \wedge C \mid C \lessdot_\sigma A,\, C \wedge R \neq O\} = \bigvee O = O$, which is a contradiction.

"$\Leftarrow$": Given $\sigma \in \mathbb{Z}$, if $C \wedge R \neq O$, for all $C \lessdot_\sigma A$, then $\rho_\sigma(A \mid R) = \bigvee\{C \mid C \lessdot_\sigma A, C \wedge R \neq O\} = \bigvee\{C \mid C \lessdot_\sigma A\} = A \Rightarrow A \in V_\sigma$, for $\sigma \in \mathbb{Z}$. Q.E.D.

In other words, in the case of infinite $\vee$-distributive lattices, we have that $A \in V_\sigma$ if and only if all σ-connected components of A are marked by the root marker R.

Consider now analysis and synthesis operators given by:

$$\psi_\sigma^\uparrow \;=\; \rho_{\sigma+1}(\cdot \mid R) \quad \text{and} \quad \psi_\sigma^\downarrow \;=\; \mathbf{id}, \quad \sigma \in \mathbb{Z}. \tag{6.75}$$

It is easy to see that, in this case, the pyramid condition $\psi_\sigma^\uparrow \psi_\sigma^\downarrow = \mathbf{id}$ on $V_{\sigma+1}$ is satisfied: $\psi_\sigma^\uparrow \psi_\sigma^\downarrow = \psi_\sigma^\uparrow = \rho_{\sigma+1}(\cdot \mid R)$, and $\rho_{\sigma+1}(\cdot \mid R)$ coincides with the identity operator on $V_{\sigma+1} = \mathrm{Inv}(\rho_{\sigma+1}(\cdot \mid R))$, since $\rho_{\sigma+1}(\cdot \mid R)$ is idempotent, for $\sigma \in \mathbb{Z}$.

Now, assume that there exist subtraction and addition operations $\dot{-}$, $\dot{+}$ defined on $\mathcal{L}$. In this case, all difference spaces coincide with $\mathcal{L}$. Given $\sigma_0 \in \mathbb{Z}$ and an arbitrary $A \in \mathcal{L}$, we have that $A_0 = \rho_{\sigma_0}(A \mid R) \in V_{\sigma_0}$, so that we can apply the pyramid transform, given by (6.71) and (6.72), to A_0. We therefore introduce an extra level "-1" to the pyramid transform to get:

$$A \to \{D_{-1}, A_0\} \to \{D_{-1}, D_0, A_1\} \to \cdots \to \{D_{-1}, D_0, \ldots, D_{m-1}, A_m\}, \tag{6.76}$$

for $m \geq 0$, with

$$\begin{cases} A_{j+1} \;=\; \rho_{\sigma_0+j+1}(A_j \mid R) \in V_{\sigma_0+j+1} \\ D_j \;=\; A_j \;\dot{-}\; A_{j+1} \in \mathcal{L} \end{cases}, \tag{6.77}$$

for $j = -1, 0, \ldots, m-1$, where $A_{-1} = A$. Under the additional condition

$$\rho_{\sigma_0}(A \mid R) \,\dot{+}\, (A \,\dot{-}\, \rho_{\sigma_0}(A \mid R)) \;=\; A, \quad A \in \mathcal{L}, \tag{6.78}$$

perfect reconstruction is possible, by means of the inverse pyramid transform:

$$A = A_m \,\dot{+} \sum_{j=-1}^{m} D_j, \tag{6.79}$$

where the summation refers to the addition $\dot{+}$.

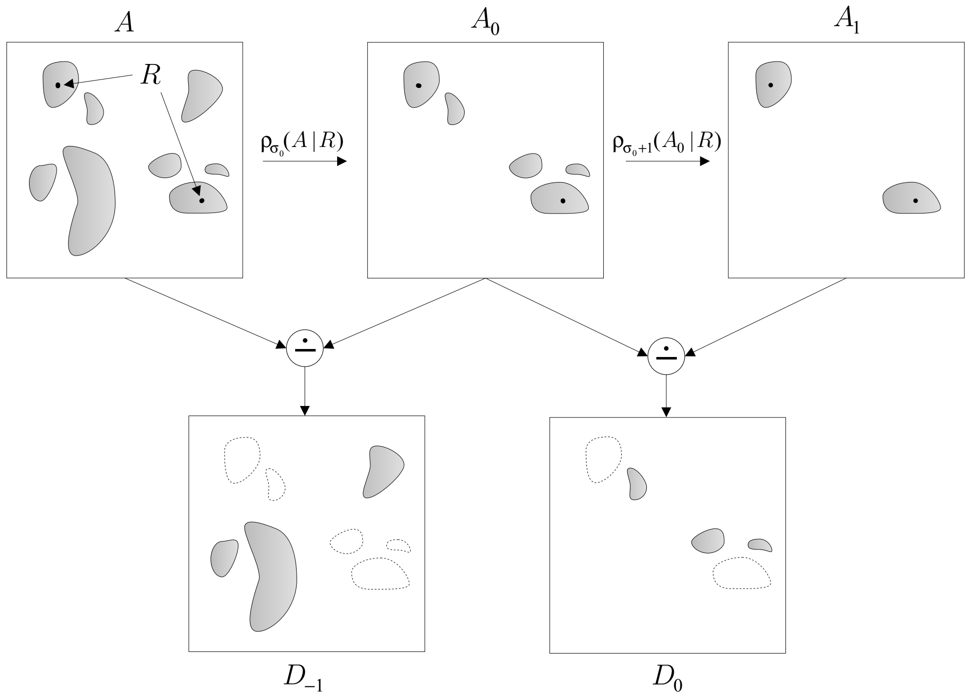


Figure 6.7: Two levels of a pyramid decomposition scheme based on the discrete dilation-pyramid multiscale connectivity of Example 6.3.15.

We remark here that the previous multiscale signal decomposition scheme is an "adjunctional pyramid," in the sense of [27]. The proposed scheme is object-based, since it does not work at the pixel level, but at the level of the connected components of an object at various scales. Note also that this scheme depends on the particular root marker R. The term "root marker" comes from the fact that the $(\sigma_0 + m)$-connected components of the original image A that are marked by R are present in all scaled signals $\{A_j \mid j = -1, 0, \ldots, m\}$ of the pyramid decomposition, which includes the basic signal A_m. Therefore, the root marker should be selected so as to preserve important objects in the image that should never go into the detail signals.

Fig. 6.7 depicts an example of the proposed multiscale signal decomposition scheme, where $\mathcal{L} = \mathcal{P}(E)$, E is a square subset of $\mathbb{R}^2$, furnished with the Euclidean topology, the addition and subtraction operations are given by union and set difference, respectively (note that the perfect reconstruction condition (6.78) is satisfied in this case) and the discrete dilation-pyramid multiscale connectivity of Example 6.3.15 is considered, with the basic structuring element being a Euclidean disk. Two levels of decomposition are consid-

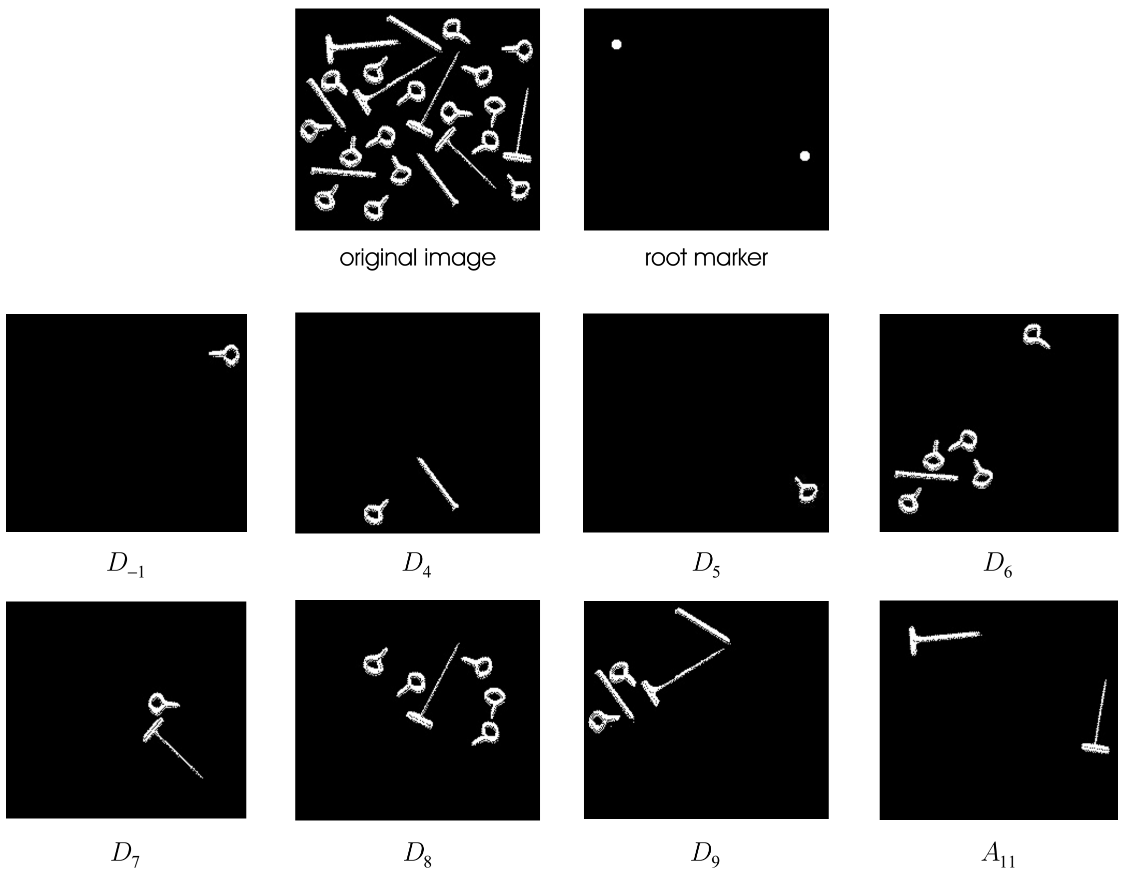


Figure 6.8: Pyramid decomposition of a real discrete image based on the discrete dilation-pyramid multiscale connectivity of Example 6.3.15.

ered; i.e., $m = 1$. Note that components marked by the root marker do not go into the detail images. Note also that the original image is recovered by the inverse pyramid transform: $A = A_1 \cup D_{-1} \cup D_0$.

Fig. 6.8 depicts an example that involves a real discrete image. Here, $\mathcal{L} = \mathcal{P}(E)$, where E is a square subset of $\mathbb{Z}^2$, the addition and subtraction operations are given by union and set difference, respectively (as in the previous example, the perfect reconstruction condition (6.78) is satisfied) and the discrete dilation-pyramid multiscale connectivity of Example 6.3.15 is considered, where the base connectivity is given by 4-adjacency connectivity and the basic structuring element is the 3×3 cross. Note that the σ-reconstructions needed to compute the pyramid decomposition can be implemented by means of (6.36). In this example, $\sigma_0 = -12$ (this is the smallest scale that generates a nonzero initial detail image D_{-1}), and twelve levels of the decomposition are considered; i.e., $m = 11$. The detail images D_0, D_1, D_2, D_3, and D_{10} are zero, and are therefore not shown. Note that the low-index detail images contain outliers, with respect to the objects selected by the root marker. In a sense, this agrees with an earlier remark that the detail images at lower scales correspond to less relevant information. Note that the original image is recovered by the inverse pyramid transform; i.e., by union of the basic and detail images shown.

6.5.2 Hierarchical Segmentation and Hierarchical Clustering

The concept of hierarchical segmentation is of fundamental importance in multiscale applications, such as adaptive bit-rate object-based coding of still images and image sequences [72]. In these applications, it is desirable to have several levels of segmentation at various scales, so that the amount of compression (bit-rate) can be adjusted to meet varying transmission/storage requirements.

On the other hand, hierarchical clustering is a technique used to group together similar objects in a hierarchical fashion, with applications in unsupervised classification algorithms, where the number, or the statistical distribution, of classes is not known a priori [23, 36]. A good analogy is provided by the field of biological taxonomy, where species are grouped into genera, genera into families, families into orders, and so on. For this reason, hierarchical clustering is part of the field of "numerical taxonomy" [84]. Hierarchical clustering, and clustering in general, is done by partitioning a feature space, in which each object is represented by a point. The availability of several levels of clustering in hierarchical clustering algorithms is often helpful in revealing the true structure of the data; e.g., the number of classes that best represent the organization of the data.

In this dissertation, we restrict our attention to *nested* hierarchical segmentation and *nested* hierarchical clustering, meaning that the several levels of segmentation or clustering must be ordered in a nested sequence, from coarse to fine. These notions of hierarchical segmentation and hierarchical clustering can be formalized by using the concept of a hierarchical partition, defined next. Recall from Section 4.1.2 the definition of a partition, and the definitions of partial order $\sqsubseteq$ and infimum $\sqcap$ in the lattice of partitions.

6.5.2 Definition. Let $\mathcal{L}$ be a lattice with sup-generating family $\mathcal{S}$. A *hierarchical partition* of an element $A \in \mathcal{L}$ is a mapping $\mathbf{p}_A$: $\mathbb{R} \times \mathcal{S}(A) \to \mathcal{L}$, such that:

(*i*) $\mathbf{p}_A(\sigma, \cdot)$ is a partition of A, for each $\sigma \in \mathbb{R}$,

(*ii*) $\mathbf{p}_A(\sigma, \cdot) \sqsubseteq \mathbf{p}_A(\tau, \cdot)$, if $\sigma \geq \tau$.

Each $\mathbf{p}_A(\sigma, x)$ is called a *σ-zone* of the hierarchical partition $\mathbf{p}_A$ of A. We say that $\mathbf{p}_A$ is *connected* (with respect to a given multiscale connectivity system $(\varphi, \mathbf{C})$ on $\mathcal{L}$) if $\mathbf{p}_A(\sigma, x) \in \mathbf{C}(\sigma)$, for $\sigma \in \mathbb{R}$ and $x \in \mathcal{S}(A)$. Moreover, we say that $\mathbf{p}_A$ is *coercive* if it satisfies the semi-continuity property $\mathbf{p}_A(\sigma, \cdot) = \sqcap_{\tau<\sigma} \mathbf{p}_A(\tau, \cdot)$, for each $\sigma \in \mathbb{R}$. △

The partitions $\mathbf{p}_A(\sigma, \cdot)$ are said to be the *σ-levels* or the *σ-partitions* of $\mathbf{p}_A$, for $\sigma \in \mathbb{R}$. The σ-partition $\mathbf{p}_A(\sigma, \cdot)$ corresponds to the partition of A at scale σ. The *nesting* property (ii) of Definition 6.5.2 requires the σ-partitions to be increasingly finer or nested. The coercivity property implies that the σ-partitions are nested *tightly*; i.e., for each $\sigma \in \mathbb{R}$, $\mathbf{p}_A(\sigma, \cdot)$ is the coarsest partition of A that is finer than all partitions $\mathbf{p}_A(\tau, \cdot)$, for $\tau < \sigma$. Coercivity is not a fundamental property of a hierarchical partition; in most applications, such as (nested) hierarchical segmentation and (nested) hierarchical clustering, nesting of the partitions is all that is required.

Next, we show that a σ-zone of a hierarchical partition equals the supremum of all τ-zones it majorates, for given $\tau \geq \sigma$. This reflects the intuitive idea that, as one goes in the direction of lower scales, the partitions get increasingly coarser through "merging" of zones at previous levels. The following result is similar to the Lemma in [79, see p. 240]; however, the proof that we give below is new.

6.5.3 Proposition. Let $\mathcal{L}$ be a lattice with sup-generating family $\mathcal{S}$, and let $\mathbf{p}_A$ be a hierarchical partition of $A \in \mathcal{L}$. For each $x \in \mathcal{S}(A)$ and $\tau \geq \sigma$,

$$\mathbf{p}_A(\sigma, x) = \bigvee_{y \in \mathcal{S}(A)} \{\mathbf{p}_A(\tau, y) \mid \mathbf{p}_A(\tau, y) \leq \mathbf{p}_A(\sigma, x)\}. \tag{6.80}$$

□

Proof. The proof of the inequality $\geq$ is trivial. We show the converse inequality. First, note that, if $y \in \mathcal{S}(A)$ but $y \not\leq \mathbf{p}_A(\sigma, x)$, then $\mathbf{p}_A(\tau, y) \leq \mathbf{p}_A(\sigma, y)$ and $\mathbf{p}_A(\sigma, y) \wedge \mathbf{p}_A(\sigma, x) = O$, which imply that $\mathbf{p}_A(\tau, y) \wedge \mathbf{p}_A(\sigma, x) = O \Rightarrow \mathbf{p}_A(\tau, y) \not\leq \mathbf{p}_A(\sigma, x)$. Therefore, the right-hand side of (6.80) reduces to $\bigvee_{y \leq \mathbf{p}_A(\sigma,x)} \{\mathbf{p}_A(\tau, y) \mid \mathbf{p}_A(\tau, y) \leq \mathbf{p}_A(\sigma, x)\}$. Let $y \leq \mathbf{p}_A(\sigma, x)$. We have that $y \leq \mathbf{p}_A(\tau, y) \leq \mathbf{p}_A(\sigma, y)$, so that $\mathbf{p}_A(\sigma, y) \wedge \mathbf{p}_A(\sigma, x) \geq y \neq O \Rightarrow \mathbf{p}_A(\sigma, y) = \mathbf{p}_A(\sigma, x)$; i.e., $\mathbf{p}_A(\tau, y) \leq \mathbf{p}_A(\sigma, x)$. Hence, $\mathbf{p}_A(\sigma, x) = \bigvee_{y \leq \mathbf{p}_A(\sigma,x)} y \leq \bigvee_{y \leq \mathbf{p}_A(\sigma,x)} \mathbf{p}_A(\tau, y) = \bigvee_{y \leq \mathbf{p}_A(\sigma,x)} \{\mathbf{p}_A(\tau, y) \mid \mathbf{p}_A(\tau, y) \leq \mathbf{p}_A(\sigma, x)\}$, as required. Q.E.D.

In Proposition 4.1.8, it was shown that the connected components of $A \in \mathcal{L}$ provide a connected partition A, which is the coarsest possible partition of A. We now provide the multiscale extension of that result. First, we need the following definition.

6.5.4 Definition. Given two hierarchical partitions $\mathbf{p}_A$ and $\mathbf{p}'_A$, we say that $\mathbf{p}_A$ is *finer* than $\mathbf{p}'_A$ if $\mathbf{p}_A(\sigma, \cdot) \sqsubseteq \mathbf{p}'_A(\sigma, \cdot)$, for each $\sigma \in \mathbb{R}$; i.e., each σ-partition of $\mathbf{p}_A$ is finer than the corresponding σ-partition of $\mathbf{p}'_A$. In this case, we also say that $\mathbf{p}'_A$ is *coarser* than $\mathbf{p}_A$. △

We remark here that this defines a partial order relation on the set of all hierarchical partitions of A, which becomes a lattice with infimum and supremum defined pointwise. We have the following proposition.

6.5.5 Proposition. Let $\mathcal{L}$ be a lattice with sup-generating family $\mathcal{S}$, furnished with a multiscale connectivity system $(\varphi, \mathbf{C})$, and let $\{\gamma_{\sigma,x} \mid \sigma \in \mathbb{R}, x \in \mathcal{S}\}$ be the σ-connectivity openings associated with $(\varphi, \mathbf{C})$.

(a) For $A \in \mathcal{L}$, the mapping $\mathbf{c}_A\colon \mathbb{R} \times \mathcal{S}(A) \to \mathcal{L}$, given by

$$\mathbf{c}_A(\sigma, x) = \gamma_{\sigma,x}(A), \quad \sigma \in \mathbb{R},\ x \in \mathcal{S}(A), \tag{6.81}$$

is a connected hierarchical partition of A. Moreover, it is the coarsest possible connected hierarchical partition of A.

(b) If the σ-connectivity openings $\{\gamma_{\sigma,x} \mid \sigma \in \mathbb{R}, x \in \mathcal{S}\}$ are $\downarrow$-continuous, then $\mathbf{c}_A$ is coercive. □

Proof. (a): It is clear that $\mathbf{c}_A(\sigma, \cdot)$ is a partition of A and that $\mathbf{c}_A(\sigma, x) \in \mathbf{C}(\sigma)$, for $\sigma \in \mathbb{R}$, $x \in \mathcal{S}(A)$. Also, the nesting property $\mathbf{c}_A(\sigma, \cdot) \sqsubseteq \mathbf{c}_A(\tau, \cdot)$, if $\sigma \geq \tau$, follows from property (ii) of Theorem 6.1.13. Finally, that $\mathbf{c}_A$ is the coarsest possible connected hierarchical partition of A follows directly from Proposition 4.1.8.

(b): This follows directly from Proposition 6.1.14. Q.E.D.

The hierarchical partition $\mathbf{c}_A$ is referred to as the *hierarchical partition of connected components* (HPCC) of A. Note that the dilation-pyramid multiscale connectivity of Example 6.3.11 and the opening-pyramid multiscale connectivity of Example 6.3.29 produce coercive HPCCs, since the σ-connectivity openings associated with these multiscale connectivities are $\downarrow$-continuous.

Fig. 6.9 depicts an image and a few σ-partitions of its HPCC. In this example, $\mathcal{L} = \mathcal{F}(E)$, where E is a square subset of $\mathbb{R}^2$, furnished with the Euclidean topology, and the dilation-pyramid multiscale connectivity of Example 6.3.11 is considered, with the basic structuring element being a Euclidean disk. Notice that, as scale increases, the σ-partitions become increasingly finer; i.e., the nesting property is satisfied.

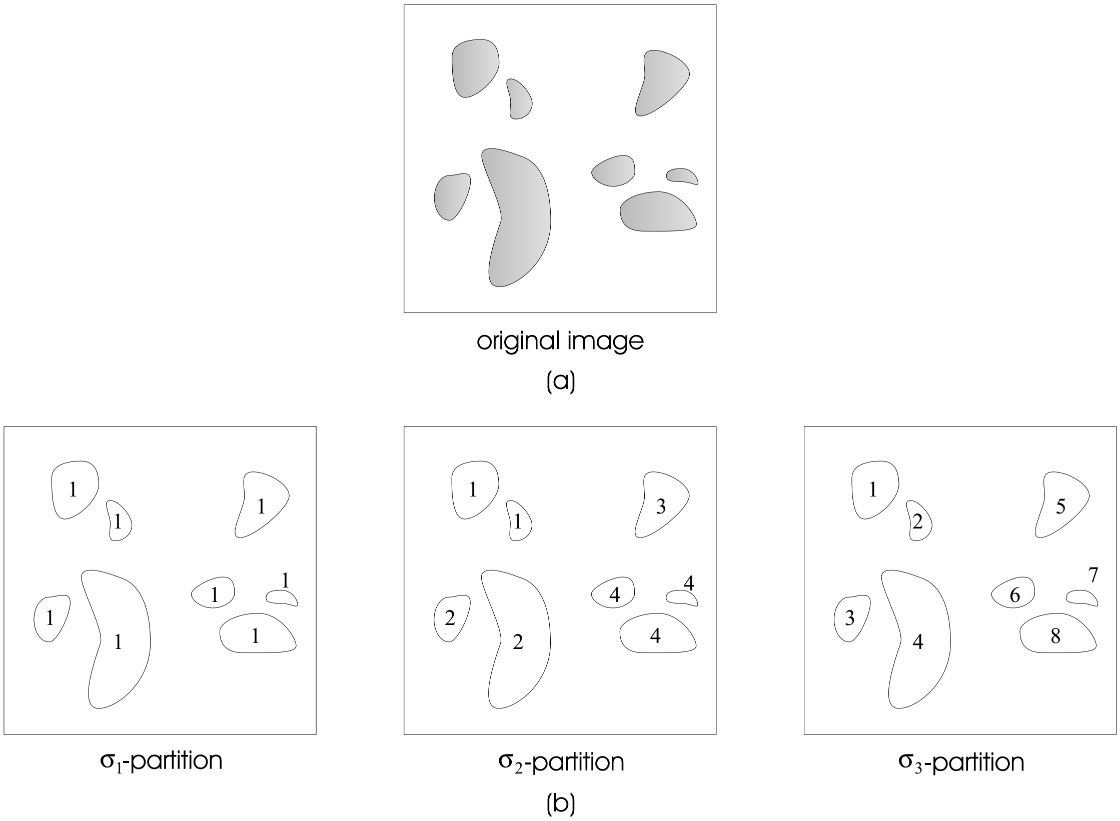


Figure 6.9: (a) Original image, and (b) three σ-partitions of its HPCC, according to the dilation-pyramid multiscale connectivity of Example 6.3.11. Here, $\sigma_1 \leq \sigma_2 \leq \sigma_3$. Note that the nesting property is satisfied.

Another interesting example of hierarchical partition derived from a multiscale connectivity is given next. Recall from Section 5.2 that, given a function $f \in \mathrm{Fun}(E, \mathcal{T})$, we can define a mapping $F : \mathcal{T} \to \mathcal{P}(E)$ given by

$$F(t) = \{x \in E \mid f(x) = t\}, \quad t \in \mathcal{T}. \tag{6.82}$$

We have the following definition.

6.5.6 Definition. Let $\mathcal{L} = \mathcal{P}(E)$, with the points as sup-generators, furnished with a multiscale connectivity system $(\varphi, \mathbf{C})$, and let $\{\gamma_{\sigma,x} \mid \sigma \in \mathbb{R}, x \in \mathcal{S}\}$ be the σ-connectivity openings associated with $(\varphi, \mathbf{C})$. For an image $f \in \mathrm{Fun}(E, \mathcal{T})$, the mapping $\mathbf{z}_f$: $\mathbb{R} \times \mathcal{S} \to \mathcal{P}(E)$, given by

$$\mathbf{z}_f(\sigma, x) = \gamma_x(F(f(x))), \quad \sigma \in \mathbb{R},\ x \in \mathcal{S}, \tag{6.83}$$

is a connected hierarchical partition of the domain of definition E, called the *hierarchical partition of flat zones* of f. △

Clearly, if the σ-connectivity openings $\{\gamma_{\sigma,x} \mid \sigma \in \mathbb{R}, x \in \mathcal{S}\}$ are $\downarrow$-continuous, then z_f is coercive. The hierarchical partition of flat zones can be applied in the hierarchical segmentation of grayscale as well as multispectral images.

The discretization of the previous concepts is straightforward. The discrete counterpart of Definition 6.5.2 is the following.

6.5.7 Definition. Let $\mathcal{L}$ be a lattice with sup-generating family $\mathcal{S}$. A *discrete hierarchical partition* of an element $A \in \mathcal{L}$ is a mapping $\mathbf{p}_A$: $\mathbb{Z} \times \mathcal{S}(A) \to \mathcal{L}$, such that:

(*i*) $\mathbf{p}_A(\sigma, \cdot)$ is a partition of A, for each $\sigma \in \mathbb{Z}$,

(*ii*) $\mathbf{p}_A(\sigma, \cdot) \sqsubseteq \mathbf{p}_A(\tau, \cdot)$, if $\sigma \geq \tau$.

Each $\mathbf{p}_A(\sigma, x)$ is called a *σ-zone* of the discrete hierarchical partition $\mathbf{p}_A$ of A. We say that $\mathbf{p}_A$ is *connected* (with respect to a given discrete multiscale connectivity system $(\varphi, \mathbf{C})$ on $\mathcal{L}$) if $\mathbf{p}_A(\sigma, x) \in \mathbf{C}(\sigma)$, for $\sigma \in \mathbb{Z}$ and $x \in \mathcal{S}(A)$. △

As before, the partitions $\mathbf{p}_A(\sigma, \cdot)$ are said to be the *σ-levels* or the *σ-partitions* of $\mathbf{p}_A$, for $\sigma \in \mathbb{Z}$. There is no discrete counterpart to the coercivity property; only the nesting property is retained in the discrete case. The discrete version of Proposition 6.5.3 is trivial. Below, we provide the discrete version of Proposition 6.5.5.

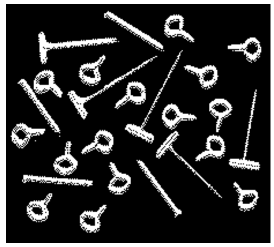
6.5.8 Proposition. Let $\mathcal{L}$ be a lattice with sup-generating family $\mathcal{S}$, furnished with a discrete multiscale connectivity system $(\varphi, \mathbf{C})$, and let $\{\gamma_{\sigma,x} \mid \sigma \in \mathbb{Z}, x \in \mathcal{S}\}$ be the σ-connectivity openings associated with $(\varphi, \mathbf{C})$. For $A \in \mathcal{L}$, the mapping $\mathbf{c}_A$: $\mathbb{Z} \times \mathcal{S}(A) \to \mathcal{L}$, given by

$$\mathbf{c}_A(\sigma, x) = \gamma_{\sigma,x}(A), \quad \sigma \in \mathbb{Z},\ x \in \mathcal{S}(A), \tag{6.84}$$

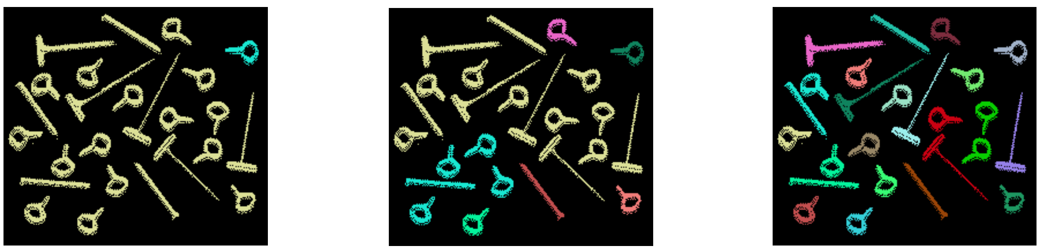
is a connected discrete hierarchical partition of A. Moreover, it is the coarsest possible connected discrete hierarchical partition of A. □

The discrete hierarchical partition $\mathbf{c}_A$ is also referred to as the discrete HPCC of A.

Fig. 6.10 depicts an application of the HPCC in hierarchical segmentation. A discrete image is depicted in (a), whereas three σ-partitions of its HPCC are depicted in (b). Different colors indicate different σ-zones. In this example, $\mathcal{L} = \mathcal{P}(E)$, where E is



original image
(a)



σ_1-partition (2 zones) | σ_2-partition (7 zones) | σ_3-partition (21 zones)

(b)

Figure 6.10: (a) Original discrete image, and (b) three σ-partitions of its HPCC, according to the discrete dilation-pyramid multiscale connectivity of Example 6.3.15. Here, $\sigma_1 = -8, \sigma_2 = -5, \sigma_3 = -2$. Note that the nesting property is satisfied.

a square subset of $\mathbb{Z}^2$, and the discrete dilation-pyramid multiscale connectivity of Example 6.3.15 is considered, where the base connectivity is given by 4-adjacency connectivity and the basic structuring element is the 3×3 cross. Notice that, due to the nesting property, as scale increases, the σ-partitions produce segmentations with more objects; i.e., some objects from previous scales are "broken apart."

On the other hand, Fig. 6.11 depicts an application of the HPCC in hierarchical clustering. In (a), a 2-D discrete feature space containing simulated data is depicted, whereas three σ-partitions of its HPCC are depicted in (b). The σ-zones, indicated with different colors, correspond to different classes detected at scale σ. A few σ-zones that contain only a small number of data points, which correspond to outliers, are not shown. Note that the classes exhibit elongated, linear distributions. In this example, $\mathcal{L} = \mathcal{P}(E)$, where E is a square subset of $\mathbb{Z}^2$, and the discrete dilation-pyramid multiscale connectivity of Example 6.3.15 is considered, where the base connectivity is given by 4-adjacency connectivity and the struc-

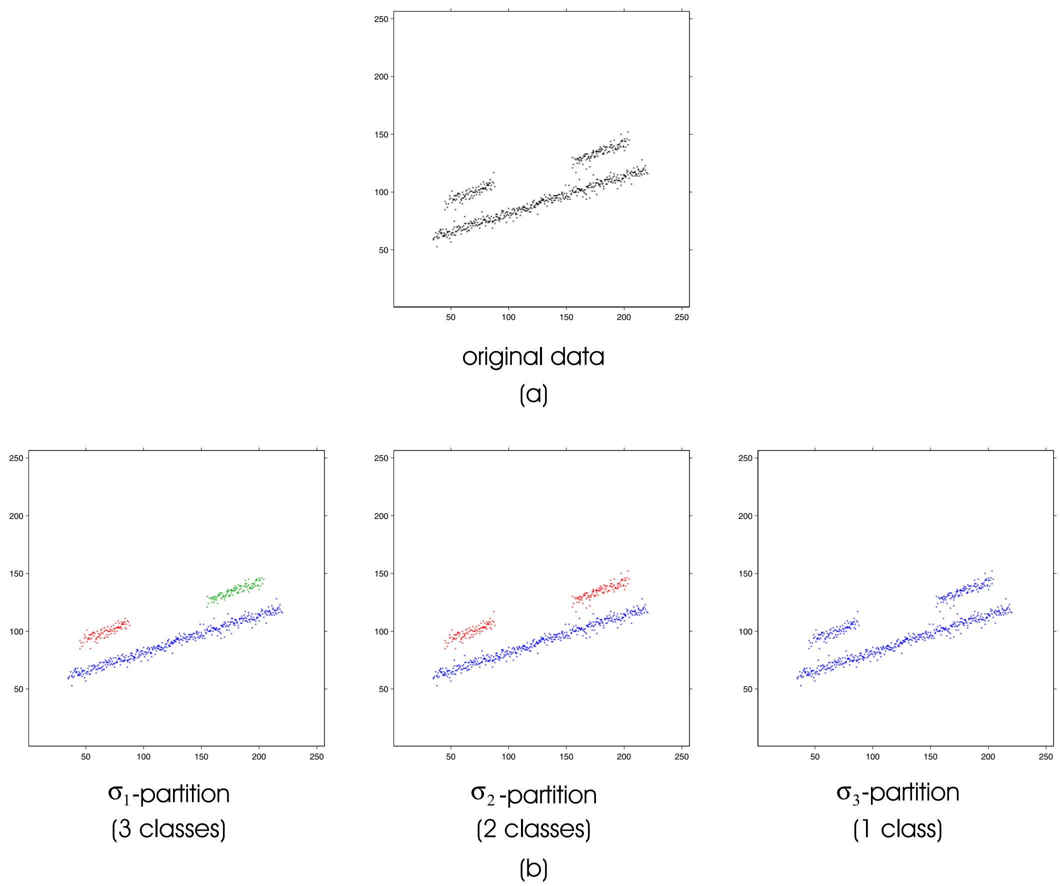


Figure 6.11: (a) Original discrete image, and (b) three σ-partitions of the HPCC, according to the discrete dilation-pyramid multiscale connectivity of Example 6.3.15; the structuring element used is a thin rectangle oriented along the direction of the clusters. Here, $\sigma_1 = -1$, $\sigma_2 = -2$, and $\sigma_3 = -3$. The classification that corresponds to the σ_2-partition is particularly hard to obtain using conventional clustering methods.

turing element is a thin rectangle oriented along the direction of the clusters (this reflects prior knowledge about the problem). At large scale, three classes are detected, whereas at low scale, the entire data are considered to constitute a single class. The classification that corresponds to the σ_2-partition is particularly hard to obtain using minimum-square distance schemes and thus implicitly assume hyperellipsoidal clusters [23, 62]. We remark that the proposed hierarchical clustering algorithm, based on HPCCs, belongs to the class of so-called *scale-space clustering algorithms* [17, 46, 62, 90, 92]. These hierarchical clustering techniques have been recently proposed and share the common feature of explicitly using scale as the free parameter governing the construction of the clustering hierarchy.

Finally, we remark that the discretization of the notion of hierarchical partition of flat zones is obvious, and thus it will not be repeated here. Note that, in this case, the function lattice $\mathrm{Fun}(E, \mathcal{T})$ can still be arbitrary; only the set of scales is discrete.

6.5.3 Multiscale Features

Image features are fundamental constituents of pattern recognition algorithms for image analysis. The performance of such algorithms is directly related to the choice of robust features. In this section, we propose multiscale image analysis features, namely, the *clustering curve* and the *clustering spectrum*, which measure the multiscale connectivity properties of a given object. We remark that, in Mathematical Morphology, a useful and well-known example of multiscale image analysis feature is the *pattern spectrum* [55]. The clustering spectrum is distinct from the pattern spectrum, since the latter is based on measurements made on a granulometric distribution, whereas the former is based on measurements made on a hierarchical partition. Nevertheless, clustering spectra and pattern spectra are similar tools and share similar properties.

A *feature* on a lattice $\mathcal{L}$ is a mapping $s\colon \mathcal{L} \to \overline{\mathbb{R}}$. In practice, a feature measures some property of elements of $\mathcal{L}$. For instance, a connectivity measure on $\mathcal{L}$ is a feature that measures the degree of connectivity of elements of $\mathcal{L}$. Another example is the *area feature* on $\mathcal{P}(\mathbb{R}^2)$, given by $s(A) = \lambda(A)$, the 2-D Lebesgue measure of A [7], if A is measurable, or $s(A) = 0$, otherwise, which measures the "area" of A (similarly, one can define a *length feature* on $\mathcal{P}(\mathbb{R})$, a *volume feature* on $\mathcal{P}(\mathbb{R}^3)$, as well as higher-dimensional analogs on $\mathcal{P}(\mathbb{R}^n)$, for $n \geq 4$). Yet another example is the *discrete area feature* on $\mathcal{P}(\mathbb{Z}^2)$, given by $s(A) = \mathrm{Card}(A)$, the cardinality of A, which measures the "discrete area" of A (accordingly, one can define a *discrete length feature* on $\mathcal{P}(\mathbb{Z})$, a *discrete volume feature* on $\mathcal{P}(\mathbb{Z}^3)$, as well as higher-dimensional analogs on $\mathcal{P}(\mathbb{Z}^n)$, for $n \geq 4$). Any remarks made below about the area feature and its discrete counterpart also applies to length, volume, and analogous higher-dimensional features, as well as to their discrete counterparts, respectively.

A feature s on $\mathcal{L}$ is *monotone* if it is either increasing or decreasing, with respect to the partial orders of $\mathcal{L}$ and $\overline{\mathbb{R}}$. Note that, in general, connectivity measures are not monotone. The area feature is not monotone either, due to the existence of nonmeasurable sets in $\mathbb{R}^2$. On the other hand, the discrete area feature is increasing. A monotone feature s on $\mathcal{L}$ is said to be *lattice upper semi-continuous* (l.u.s.c.) if, for any totally ordered subset $\mathcal{Q} \subseteq \mathcal{L}$, one has that $s(\bigwedge \mathcal{Q}) = \bigwedge_{A \in \mathcal{Q}} s(A)$, if s is increasing, or $s(\bigwedge \mathcal{Q}) = \bigvee_{A \in \mathcal{Q}} s(A)$, if s is decreasing. A *lattice lower semi-continuous* (l.l.s.c.) feature is defined analogously. It is possible to show that the discrete area feature is l.l.s.c., but not l.u.s.c. For a counterexample that shows that the area feature is not l.u.s.c., let $\mathcal{Q} = \{[n, \infty) \times [n, \infty) \mid n \in \mathbb{Z}\}$. Clearly, $s(A) = \infty$, for all $A \in \mathcal{Q}$, so that $\bigwedge_{A \in \mathcal{Q}} s(A) = \infty \neq s(\bigcap \mathcal{Q}) = s(\emptyset) = 0$. We remark that any increasing feature that takes on discrete values in $\overline{\mathbb{R}} \setminus \{-\infty\}$ (resp. $\overline{\mathbb{R}} \setminus \{\infty\}$) is l.l.s.c. (resp. l.u.s.c.); a similar statement holds for decreasing features.

Next, we define the notion of a partition feature. Given a lattice $\mathcal{L}$ and $A \in \mathcal{L}$, recall from Section 4.1.2 that the set $\mathcal{P}_A$ of all partitions of A is a lattice.

6.5.9 Definition. Given a lattice $\mathcal{L}$ and $A \in \mathcal{L}$, a *partition feature* is a feature on the lattice of partitions $\mathcal{P}_A$. △

All definitions regarding features apply to partition features as well. For instance, a partition feature ν on $\mathcal{P}_A$ is increasing if $p_A \sqsubseteq p'_A \Rightarrow \nu(p_A) \leq \nu(p'_A)$, whereas it is decreasing if $p_A \sqsubseteq p'_A \Rightarrow \nu(p_A) \geq \nu(p'_A)$. A monotone partition feature ν is l.u.s.c. if, for any totally ordered subset $\mathcal{Q} \subseteq \mathcal{P}_A$, we have that $\nu(\sqcap \mathcal{Q}) = \bigwedge_{p_A \in \mathcal{Q}} \nu(p_A)$, if ν is increasing, or $\nu(\sqcap \mathcal{Q}) = \bigvee_{p_A \in \mathcal{Q}} \nu(p_A)$, if ν is decreasing. An l.l.s.c. partition feature satisfies analogous properties.

Below, we give a few examples of partition features.

6.5.10 Example.

(a) Let $\mathcal{L}$ be a lattice. Given $A \in \mathcal{L}$, a useful partition feature on $\mathcal{P}_A$ is given by:

$$\mu(p_A) = \text{cardinality of the set of zones of } p_A, \tag{6.85}$$

for $p_A \in \mathcal{P}_A$. We refer to μ as the *counting feature*. This partition feature is clearly decreasing.

(b) Let s be a feature on a lattice $\mathcal{L}$. Given $A \in \mathcal{L}$, consider the following three partition features on $\mathcal{P}_A$:

$$\nu_{\text{avg}}(p_A) = \begin{cases} s(A)/\mu(A), & \text{if } s(A) \text{ or } \mu(A) \text{ is finite} \\ 0 & \text{otherwise} \end{cases}, \tag{6.86}$$

$$\nu_{\text{inf}}(p_A) = \bigwedge_{x \in \mathcal{S}(A)} s(p_A(x)), \tag{6.87}$$

$$\nu_{\text{sup}}(p_A) = \bigvee_{x \in \mathcal{S}(A)} s(p_A(x)), \tag{6.88}$$

for $p_A \in \mathcal{P}_A$. The partition feature ν_{avg} is always increasing, whereas ν_{inf} and ν_{sup} are increasing or decreasing if the feature s has these properties, respectively. $\lozenge$

As a concrete instance of Example 6.5.10(b), consider the case when s is the area feature on $\mathcal{P}(\mathbb{R}^2)$. In this case, ν_{avg} is an increasing feature that gives the "average" zone area of a partition (if A and all zones are measurable), ν_{inf} gives the "smallest" zone area, and ν_{sup} gives the "largest" zone area; ν_{inf} and ν_{sup} are neither increasing or decreasing. Similar remarks apply when s is the discrete area feature on $\mathcal{P}(\mathbb{Z}^2)$. However, unlike the previous case, the discrete area feature is increasing. Hence, in addition to ν_{avg}, we have that ν_{inf} and ν_{sup} are increasing partition features as well.

We now have the following result.

6.5.11 Proposition. Let s be a feature on a lattice $\mathcal{L}$, and let $A \in \mathcal{L}$.

(a) The counting feature μ, defined by (6.85), is l.u.s.c.

(b) Assume that s is finite. The partition feature ν_{avg}, defined by (6.86), is l.u.s.c.

(c) Assume that s is increasing. If s is l.u.s.c., the partition feature ν_{inf}, defined by (6.87), is l.u.s.c. as well.

(d) Assume that s is decreasing. If s is l.l.s.c., the partition feature ν_{sup}, defined by (6.88), is l.l.s.c. as well. $\square$

PROOF. (a): Consider a totally ordered subset $\mathcal{Q} \subseteq \mathcal{P}_A$. Note that, for all $p_A \in \mathcal{Q}$, we have that $\sqcap\mathcal{Q} \sqsubseteq p_A \Rightarrow \mu(\sqcap\mathcal{Q}) \geq \mu(p_A)$, since μ is decreasing. Hence, if $\bigvee_{p_A\in\mathcal{Q}} \mu(p_A) = \infty$, then $\mu(\sqcap\mathcal{Q}) \geq \bigvee_{p_A\in\mathcal{Q}} \mu(p_A) = \infty \Rightarrow \mu(\sqcap\mathcal{Q}) = \bigvee_{p_A\in\mathcal{Q}} \mu(p_A) = \infty$, and we are done. Thus, assume that $m = \bigvee_{p_A\in\mathcal{Q}} \mu(p_A)$ is finite. In this case, we must have $m = \mu(p_A^0)$, for some $p_A^0 \in \mathcal{Q}$, otherwise $m-1$ would be an upper bound of the set $\{\mu(p_A) \mid p_A \in \mathcal{Q}\}$, contradicting the fact that m is its supremum. But, since $\mathcal{Q}$ is a totally ordered subset, this means that $p_A^0 \sqsubseteq p_A$, for all $p_A \in \mathcal{Q}$; i.e., $p_A^0 = \sqcap\mathcal{Q}$. Hence, $\mu(\sqcap\mathcal{Q}) = \mu(p_A^0) = m = \bigvee_{p_A\in\mathcal{Q}} \mu(p_A)$, as required.

(b): This is an easy consequence of (6.86) and part (a).

(c): Consider a totally ordered subset $\mathcal{Q} \subseteq \mathcal{P}_A$. Note that $(\sqcap\mathcal{Q})(x) = \bigwedge_{p_A\in\mathcal{Q}} p_A(x)$, for $x \in \mathcal{S}(A)$. Hence, $\nu_{\text{inf}}(\sqcap\mathcal{Q}) = \bigwedge_{x\in\mathcal{S}(A)} s(\bigwedge_{p_A\in\mathcal{Q}} p_A(x)) = \bigwedge_{x\in\mathcal{S}(A)} \bigwedge_{p_A\in\mathcal{Q}} s(p_A(x)) = \bigwedge_{p_A\in\mathcal{Q}} \bigwedge_{x\in\mathcal{S}(A)} s(p_A(x)) = \bigwedge_{p_A\in\mathcal{Q}} \nu_{\text{inf}}(p_A)$, as required.

(d): The proof is completely analogous to the proof of part (c). Q.E.D.

We remark that the result in part (a) of the previous proposition can be easily generalized in the following way. Any decreasing partition feature that takes on discrete values in $\overline{\mathbb{R}} \setminus \{-\infty\}$ (resp. $\overline{\mathbb{R}} \setminus \{\infty\}$) is l.u.s.c. (resp. l.l.s.c.); an analogous statement holds for increasing partition features. Note that the counting feature (or any decreasing partition feature that takes on discrete values, including ∞) *is not* l.l.s.c. However, one can show that, for a totally ordered subset $\mathcal{Q}$ of $\mathcal{P}_A$, we have that $\mu(\sqcup \mathcal{Q}) = \bigwedge\{\mu(p_A) \mid p_A \in \mathcal{Q}\}$, provided that $\mu(p_A) < \infty$, for some $p_A \in \mathcal{Q}$ (the proof of this fact is similar to the proof of part (a) of Proposition 6.5.11). An analogous statement applies to an increasing partition feature that takes on discrete values, including $-\infty$.

Next, we define the concept of a clustering curve.

6.5.12 Definition. Let $\mathcal{L}$ be a lattice with sup-generating family $\mathcal{S}$, furnished with a multiscale connectivity system $(\varphi, \mathbf{C})$. Given $A \in \mathcal{L}$, let ν be a partition feature on $\mathcal{P}_A$. The *clustering curve* of A with respect to ν is a function $X_A\colon \mathbb{R} \to \overline{\mathbb{R}}$, given by:

$$X_A(\sigma) = \nu(\mathbf{c}_A(\sigma, \cdot)), \quad \sigma \in \mathbb{R}, \tag{6.89}$$

where $\mathbf{c}_A$ is the HPCC of A, according to $(\varphi, \mathbf{C})$. △

Therefore, X_A indicates how the property measured by the underlying partition feature on the σ-connected components of A varies with scale. For instance, if $\nu = \mu$, the counting feature defined in (6.85), then X_A gives the number of σ-grains of A as a function of scale (in the binary case — e.g., when $\mathcal{L} = \mathcal{P}(E)$ — other useful cases include X_{A^c}, which gives the variation of the number of σ-*pores* of A, and $X_A - X_{A^c}$, which gives the variation of the σ-*genus* of A — i.e., the difference between the number of σ-grains and σ-pores). On the other hand, if $\mathcal{L} = \mathcal{P}(\mathbb{R}^2)$ and $\nu = \nu_{\text{avg}}$, the partition feature defined in (6.86), with s being the area feature, then X_A gives the average area of the σ-connected components as a function of scale. Similar remarks apply if ν is one of the partition features defined in (6.87) or (6.88), or any other partition feature. The term "clustering curve" comes from the nesting property of the HPCC, which implies that the σ-connected components of A cluster together, if scale decreases, or clusters of σ-connected components are broken apart, if scale increases. This behavior is captured by the graph of the clustering curve X_A.

If ν is an arbitrary partition feature, little can be said about the behavior of the clustering curve X_A. However, in the case of a monotone partition feature, X_A can be characterized as follows.

6.5.13 Proposition. Let $\mathcal{L}$ be a lattice, furnished with a multiscale connectivity system $(\varphi, \mathbf{C})$. Let $A \in \mathcal{L}$ and ν be a monotone partition feature on $\mathcal{P}_A$. The clustering curve $X_A\colon \mathbb{R} \to \overline{\mathbb{R}}$, defined by (6.89), satisfies the following three properties:

(a) X_A is monotone; it is increasing (resp. decreasing) if ν is decreasing (resp. increasing).

(b) X_A is continuous, except for a countable number of jump discontinuities.

(c) If the HPCC $\mathbf{c}_A$ is coercive and ν is l.u.s.c., then X_A is left-continuous at the jumps. □

Proof. (a): From the nesting property of hierarchical partitions, we have that $\mathbf{p}_A(\sigma, \cdot) \sqsubseteq \mathbf{p}_A(\tau, \cdot)$, if $\sigma \geq \tau$. If ν is increasing, then $X_A(\sigma) = \nu(\mathbf{p}_A(\sigma, \cdot)) \leq \nu(\mathbf{p}_A(\tau, \cdot)) = X_A(\tau)$, if $\sigma \geq \tau$; i.e., X_A is decreasing. The opposite holds if ν is decreasing. In either case, X_A is a monotone function.

(b): From part (a), X_A is a monotone function. If X_A is increasing (resp. decreasing), the left and right lateral limits at a point $\sigma \in \mathbb{R}$ are given, respectively, by $X_A(\sigma_-) = \lim_{\tau \uparrow \sigma} X_A(\tau) = \bigvee_{\tau<\sigma} X_A(\tau)$ (resp. $\bigwedge_{\tau>\sigma} X_A(\tau)$) and $X_A(\sigma_+) = \lim_{\tau \downarrow \sigma} X_A(\tau) = \bigwedge_{\tau>\sigma} X_A(\tau)$ (resp. $\bigvee_{\tau<\sigma} X_A(\tau)$). We conclude that the lateral limits always exist at all points $\sigma \in \mathbb{R}$. Therefore, any discontinuity of X_A must be a jump discontinuity; i.e., a point $\sigma \in \mathbb{R}$ at which the lateral limits are different, $X_A(\sigma_-) \neq X_A(\sigma_+)$. Let D be the set of such discontinuities. Since X_A is monotone, each $\sigma \in D$ is associated with a distinct open interval I_σ with endpoints at $X_A(\sigma_-), X_A(\sigma_+)$; moreover, these intervals are disjoint. Therefore, each interval contains a rational number that is not contained in the other intervals. This means that one can construct an injective function from D into $\mathbb{Q}$, the set of all rational numbers. But $\mathbb{Q}$ is a countable set, so that D must be countable as well.

(c): We show that X_A is left-continuous at any given $\sigma \in \mathbb{R}$. Since $\mathbf{c}_A$ is coercive, we have $\mathbf{c}_A(\sigma, \cdot) = \sqcap_{\tau<\sigma} \mathbf{c}_A(\tau, \cdot)$. Suppose that ν is decreasing, so that X_A is increasing. Since ν is l.u.s.c., we have $X_A(\sigma) = \nu(\mathbf{c}_A(\sigma, \cdot)) = \nu(\sqcap_{\tau<\sigma} \mathbf{c}_A(\tau, \cdot)) = \bigvee_{\tau<\sigma} \nu(\mathbf{c}_A(\tau, \cdot)) = \bigvee_{\tau<\sigma} X_A(\tau) = \lim_{\tau \uparrow \sigma} X_A(\tau) = X_A(\sigma_-)$, which shows the desired result. The case in which ν is increasing is analogous. Q.E.D.

As we remarked in Section 6.5.2, the dilation-pyramid multiscale connectivity of Example 6.3.11 and the opening-pyramid multiscale connectivity of Example 6.3.29 produce coercive HPCCs. In this case, according to part (c) of the previous proposition, left-continuity of X_A depends only on whether ν is l.u.s.c. We also remark that the definition of X_A

in (6.89) and Proposition 6.5.13 are still valid if $\mathbf{c}_A$ is replaced by any given hierarchical partition $\mathbf{p}_A$, even though one loses the interpretation of measuring the clustering of σ-connected components.

Variation in X_A indicates how the property measured by the underlying partition feature changes as the σ-connected components of A cluster together or break apart, due to change in scale. It is therefore natural to measure this variation in terms of (the absolute value of) a derivative. This leads to the definition of the clustering spectrum.

6.5.14 Definition. Let $\mathcal{L}$ be a lattice with sup-generating family $\mathcal{S}$, furnished with a multiscale connectivity system $(\varphi, \mathbf{C})$. Given $A \in \mathcal{L}$, let ν be a partition feature on $\mathcal{P}_A$. The *clustering spectrum* of A with respect to ν is a function Y_A: $\mathbb{R} \to \overline{\mathbb{R}}$, given by:

$$Y_A(\sigma) = \left| \frac{d}{d\sigma} X_A(\sigma) \right| = \left| \frac{d}{d\sigma} \nu(\mathbf{c}_A(\sigma, \cdot)) \right|, \quad \sigma \in \mathbb{R}, \tag{6.90}$$

where X_A is the clustering curve of A, defined in (6.89). △

It is understood that, at a point σ where X_A has a jump discontinuity, $Y_A(\sigma)$ will be assigned an impulse of magnitude equal to the jump. The clustering spectrum $Y_A(\sigma)$ summarizes the variation in the clustering structure of an object A as a function of scale σ. Note that, when X_A is increasing, the absolute value is redundant in (6.90).

Fig. 6.12 depicts two images as well as the clustering curve and the clustering spectrum associated with each of them. In this example, $\mathcal{L} = \mathcal{F}(E)$, where E is a square subset of $\mathbb{R}^2$, furnished with the Euclidean topology, and the dilation-pyramid multiscale connectivity of Example 6.3.11 is considered, with the basic structuring element being a Euclidean disk. The partition feature considered in this example is the counting feature μ, defined in (6.85). Since, in this case, the HPCCs are coercive and μ is l.u.s.c. (see Proposition 6.5.11), the clustering curve X_A is increasing and has a countable number of jump discontinuities, at which it is left-continuous (see Proposition 6.5.13). Moreover, X_A is piecewise constant, since μ takes on discrete values. Hence, X_A is a staircase-like function, and the clustering spectrum Y_A is thus comprised of impulses, located at the jumps of X_A. Note also that Y_A is zero over all positive scales, due to the "negative nature" of the dilation-pyramid multiscale connectivity. The clustering curves and the clustering spectra of the images in this example are very distinct, clearly indicating differences in the organization of clusters. In particular, for image A_1, the presence of components of the clustering spectrum away

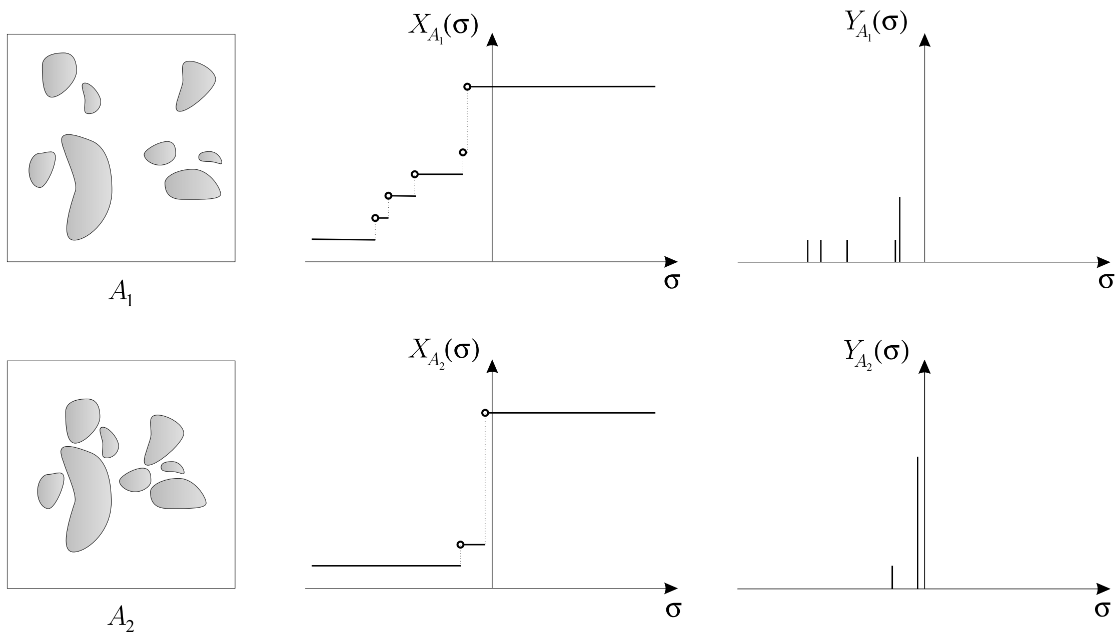


Figure 6.12: Clustering curves and spectra associated with two distinct images.

from the origin indicates existence of isolated and clearly defined clusters, as opposed to the case of image A_2.

By using the notion of discrete HPCC, we define below the concept of a discrete clustering curve.

6.5.15 Definition. Let $\mathcal{L}$ be a lattice with sup-generating family $\mathcal{S}$, furnished with a discrete multiscale connectivity system $(\varphi, \mathbf{C})$. Given $A \in \mathcal{L}$, let ν be a partition feature on $\mathcal{P}_A$. The *discrete clustering curve* of A with respect to ν is a function X_A: $\mathbb{Z} \to \overline{\mathbb{R}}$, given by:

$$X_A(\sigma) = \nu(\mathbf{c}_A(\sigma, \cdot)), \quad \sigma \in \mathbb{Z}, \tag{6.91}$$

where $\mathbf{c}_A$ is the discrete HPCC of A, according to $(\varphi, \mathbf{C})$. $\triangle$

Generally, the same remarks made previously on clustering curves apply to the discrete case as well. The discrete counterpart of the clustering spectrum is given by (the absolute value of) a "discrete derivative." We have the following definition.

6.5.16 Definition. Let $\mathcal{L}$ be a lattice with sup-generating family $\mathcal{S}$, furnished with a discrete multiscale connectivity system $(\varphi, \mathbf{C})$. Given $A \in \mathcal{L}$, let ν be a partition feature on $\mathcal{P}_A$. The *discrete clustering spectrum* of A with respect to ν is a function Y_A: $\mathbb{Z} \to \overline{\mathbb{R}}$, given by:

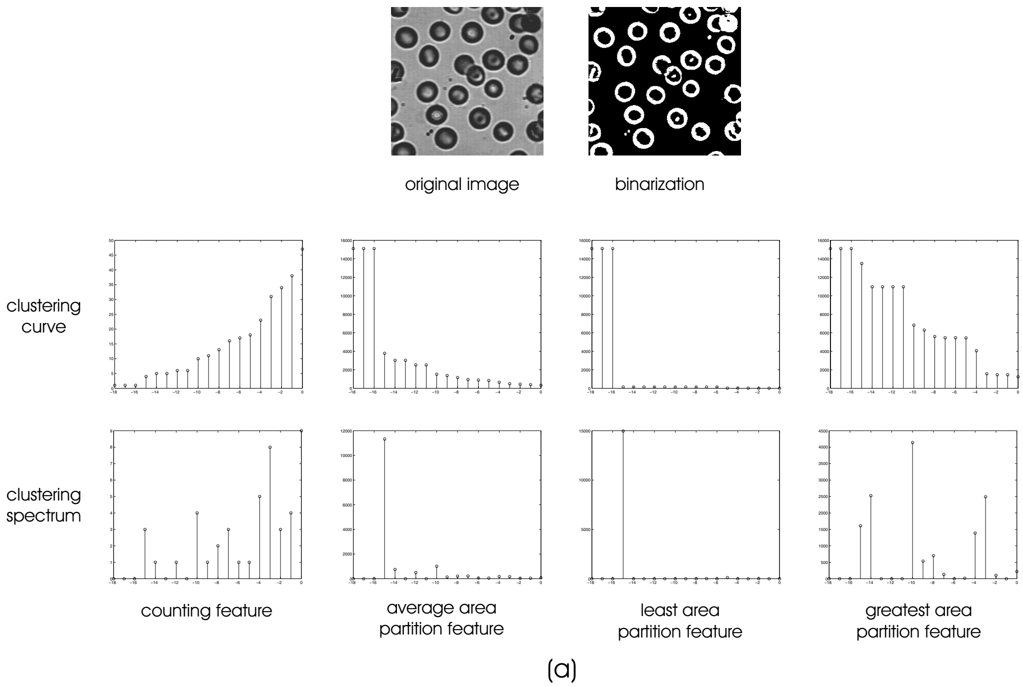
$$Y_A(\sigma) = |X_A(\sigma) - X_A(\sigma - 1)| = |\,\nu(\mathbf{c}_A(\sigma, \cdot)) - \nu(\mathbf{c}_A(\sigma - 1, \cdot))\,|, \quad \sigma \in \mathbb{Z}, \tag{6.92}$$

where X_A is the discrete clustering curve of A, defined in (6.91). △

The discrete clustering spectrum $Y_A(\sigma)$ summarizes variation in the clustering structure of an object A as a function of scale σ. Note that, when X_A is increasing, the absolute value is redundant in (6.92).

Fig. 6.13 depicts two real discrete images and their discrete clustering curves and spectra corresponding to four different partition features: the counting feature μ, the average area feature ν_{avg}, the least area feature ν_{inf}, and the greatest area feature ν_{sup}. In this example, $\mathcal{L} = \mathcal{P}(E)$, where E is a square subset of $\mathbb{Z}^2$, and the discrete dilation-pyramid multiscale connectivity of Example 6.3.15 is considered, where the base connectivity is given by 4-adjacency connectivity and the basic structuring element is a 3×3 cross. Only the negative scales are displayed (as in the previous example, the clustering spectrum is zero over the positive scales). In particular, the origin is located on the right hand side of the plots. Note that the clustering curves associated with the counting feature are increasing, whereas the clustering curves associated with all other partition features are decreasing, as expected. Note also that the blood cell image in (a) contains more clearly defined cell clusters than the bone marrow cell image in (b). Accordingly, the clustering spectra of the blood cell image contain components, away from the origin, of larger magnitude than the corresponding clustering spectra of the bone marrow cell image. Finally, we also observe that the least area partition feature produces clustering curves and clustering spectra that contain less information than the ones produced by the other partition features.

Another application of the clustering spectrum is depicted in Fig. 6.14. The objective here is to detect *directional* clustering in an image. The simulated discrete image depicted in Fig. 6.14 contains objects that form clusters in a particular direction (for instance, this situation can happen in an aerial image of a highway, or a surveillance image of a production line). In this example, $\mathcal{L} = \mathcal{P}(E)$, where E is a square subset of $\mathbb{Z}^2$, and the discrete dilation-pyramid multiscale connectivity of Example 6.3.15 is considered, where the base connectivity is given by 4-adjacency connectivity. Two small rectangles are employed as basic structuring elements; one is oriented along the direction of clustering, whereas the



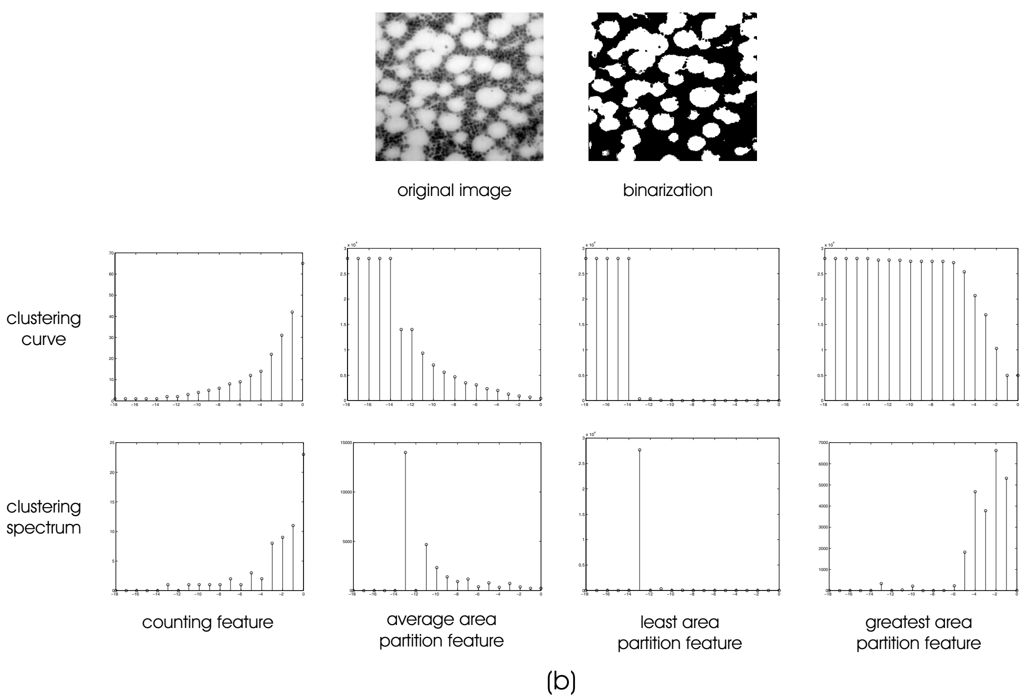


Figure 6.13: Clustering curves and spectra associated with two distinct images, according to the discrete dilation-pyramid multiscale connectivity of Example 6.3.15: (a) blood cell image, and (b) bone marrow cell image.

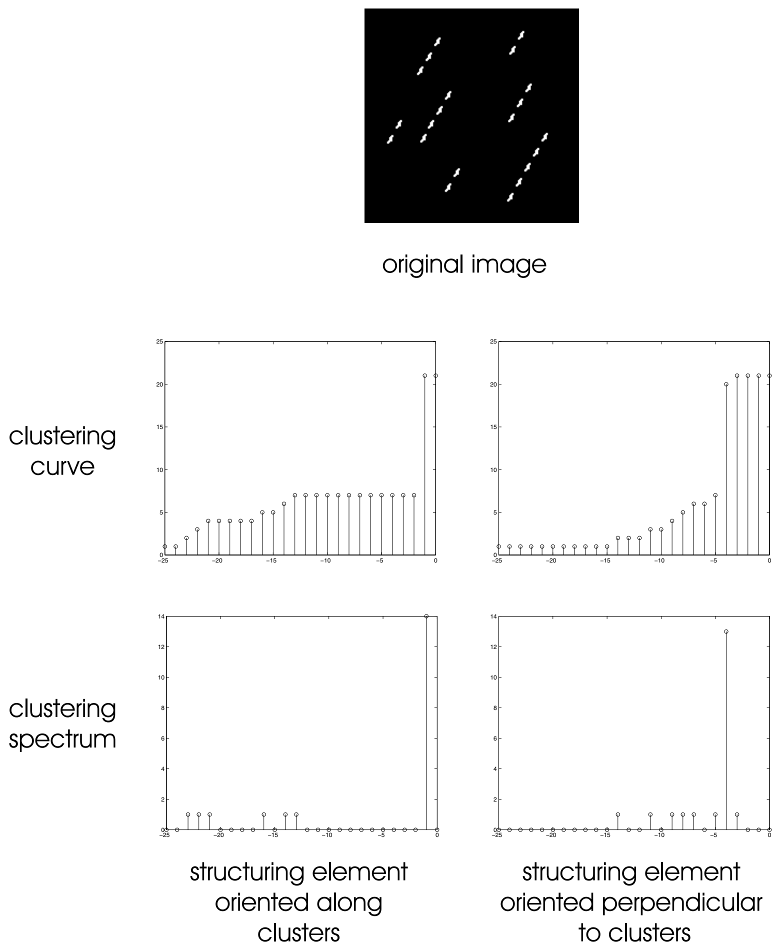


Figure 6.14: Clustering curves and spectra associated with a binary image containing objects clustered directionally, according to the discrete dilation-pyramid multiscale connectivity of Example 6.3.15.

other is oriented perpendicular to it. Once again, only the negative scales are displayed, so that the origin is located on the right hand side of the plots. The underlying partition feature used is the counting feature. The presence of significant components in the first clustering spectrum that are *close* to the origin indicate that the objects are clustered along the direction of the first structuring element.

6.6 Multiscale Hyperconnectivity

In this section, we extend the concept of multiscale connectivity by introducing the notion of multiscale hyperconnectivity. Like its single-scale counterpart, multiscale hyperconnectivity allows for hyperconnected components that have nonzero infimum.

We restrict ourselves to the discrete-scale case. Hence, the theory of multiscale hyperconnectivity developed here is analogous to the theory of discrete multiscale connectivity. For the most part, it is a trivial matter to extend the definitions and results given below to the continuous-scale case.

In order to establish an axiomatization of multiscale hyperconnectivity, we introduce the notion of an overlap measure for families in a lattice $\mathcal{L}$.

6.6.1 Definition. Let $\mathcal{L}$ be a lattice. An *overlap measure* on $\mathcal{L}$ is a *decreasing* mapping $\perp\colon \mathcal{P}(\mathcal{L}) \to \overline{\mathbb{Z}}$, i.e.:

$$\mathcal{A} \subseteq \mathcal{B} \;\Rightarrow\; \perp(\mathcal{A}) \geq \perp(\mathcal{B}). \tag{6.93}$$

△

We interpret $\perp(\mathcal{A})$ as the extent to which the family $\mathcal{A}$ overlaps. Hence, $\mathcal{A}$ is said to be *fully overlapping* if $\perp(\mathcal{A}) = \infty$, whereas it is said to be *fully non-overlapping* if $\perp(\mathcal{A}) = -\infty$. The condition expressed by (6.93) reflects the observation that the degree to which a family overlaps cannot possibly be increased by adding more elements to the family.

We remark that overlap criteria, defined in Section 4.4, correspond to binary overlap measures; e.g., an overlap measure that takes on values $-\infty$ and ∞. In this sense, overlap measures are multiscale extensions of overlap criteria. In addition, given an overlap measure $\perp$ on $\mathcal{L}$, one can define a family of overlap criteria $\{\perp_\sigma\colon \mathcal{P}(\mathcal{L}) \to \{-\infty, \infty\} \mid \sigma \in \mathbb{Z}\}$ on $\mathcal{L}$, given by

$$\perp_\sigma(\mathcal{A}) = \begin{cases} \infty, & \text{if } \perp(\mathcal{A}) \geq \sigma \\ -\infty, & \text{otherwise} \end{cases}. \tag{6.94}$$

These are called the *σ-overlap criteria* associated with the overlap measure $\perp$.

We now define the concept of a hyperconnectivity measure on a lattice $\mathcal{L}$.

6.6.2 Definition. Let $\mathcal{L}$ be a lattice with sup-generating family $\mathcal{S}$, furnished with an overlap measure $\perp$. A function $\varphi_h\colon \mathcal{L} \to \overline{\mathbb{Z}}$ is said to be a *hyperconnectivity measure* on $\mathcal{L}$ if:

(*i*) $\varphi_h(O) = \varphi_h(x) = \infty$, for $x \in \mathcal{S}$,

(*ii*) for a family $\{A_\alpha\}$ in $\mathcal{L}$, we have that $\varphi_h(\bigvee A_\alpha) \geq \perp(\{A_\alpha\}) \wedge \bigwedge \varphi_h(A_\alpha)$. △

Given $A \in \mathcal{L}$, the quantity $\varphi_h(A)$ indicates the degree of hyperconnectivity of A. If $\varphi_h(A) = \infty$, A is said to be *fully hyperconnected*, whereas if $\varphi_h(A) = -\infty$, A is said to be *fully non-hyperconnected*. Intermediate hyperconnectivity, or *σ-hyperconnectivity*, is defined by saying that A is σ-hyperconnected if $\varphi_h(A) \geq \sigma$, for $\sigma \in \mathbb{Z}$. Of course, if $\sigma \geq \tau$, then σ-hyperconnectivity implies τ-hyperconnectivity.

Axiom (i) of Definition 6.6.2 requires the least element and the sup-generators to be fully hyperconnected. On the other hand, axiom (ii) requires that the degree of hyperconnectivity of the supremum of any family in $\mathcal{L}$ must always be greater than the degree of overlapping of the family *or* the least degree of hyperconnectivity of the individual elements. This implies that, in contrast to multiscale connectivities, the degree of hyperconnectivity of the supremum of a family in $\mathcal{L}$ is allowed to be less than the least degree of hyperconnectivity of the individual elements, as long as it is greater than the degree of overlapping of the family. This affords extra flexibility to the multiscale hyperconnectivity framework.

Given a hyperconnectivity class $\mathcal{H}$ on $\mathcal{L}$, furnished with an overlap criterion $\perp$, we can define a simple binary hyperconnectivity measure φ_H on $\mathcal{L}$, by letting $\varphi_H(A) = \infty$, if $A \in \mathcal{H}$, and $\varphi_H(A) = -\infty$, if $A \notin \mathcal{H}$, where the underlying overlap measure is given by the mapping $\perp$ itself. In other words, each $A \in \mathcal{L}$ is either fully hyperconnected, if $A \in \mathcal{H}$, or fully non-hyperconnected, if $A \notin \mathcal{H}$. Hence, hyperconnectivity classes lead to *single-scale* hyperconnectivities, where the degree of hyperconnectivity is all-or-nothing; i.e., there is no intermediate hyperconnectivity.

It is clear that hyperconnectivity measures generalize the notion of discrete connectivity measures; a connectivity measure φ on $\mathcal{L}$ may be considered to be a hyperconnectivity measure with respect to the "standard" overlap measure $\perp_\wedge$ given by

$$\perp_\wedge(\mathcal{A}) = \begin{cases} \infty, & \text{if } \bigwedge \mathcal{A} \neq O \\ -\infty, & \text{otherwise} \end{cases} . \tag{6.95}$$

We now define the concept of a hyperconnectivity pyramid on a lattice $\mathcal{L}$.

6.6.3 Definition. Let $\mathcal{L}$ be a lattice with sup-generating family $\mathcal{S}$, furnished with an overlap measure $\perp$. A *hyperconnectivity pyramid* on $\mathcal{L}$ is a mapping $\mathbf{H} : \mathbb{Z} \to \mathcal{P}(\mathcal{L})$ such that

(i) $\mathbf{H}(\sigma)$ is a hyperconnectivity class in $\mathcal{L}$, according to the σ-overlap criterion $\perp_\sigma$, defined by (6.94), for each $\sigma \in \mathbb{Z}$.

(ii) $\mathbf{H}(\sigma) \subseteq \mathbf{H}(\tau)$, if $\sigma \geq \tau$. △

The hyperconnectivity classes $\mathbf{H}(\sigma)$ are referred to as the *σ-levels* or the *σ-hyperconnectivity classes* associated with $\mathbf{H}$, for $\sigma \in \mathbb{Z}$. The σ-hyperconnectivity class

$\mathbf{H}(\sigma)$ corresponds to hyperconnectivity at scale σ. For $A \in \mathcal{L}$, if A is hyperconnected at all scales, i.e., if $A \in \bigcap_{\sigma \in \mathbb{Z}} \mathbf{H}(\sigma)$, A is said to be fully hyperconnected, whereas if A is not hyperconnected at any scale, i.e., if $A \notin \bigcup_{\sigma \in \mathbb{Z}} \mathbf{H}(\sigma)$, A is said to be fully non-hyperconnected. In addition, A is said to be σ-hyperconnected if $A \in \mathbf{H}(\sigma)$, for $\sigma \in \mathbb{Z}$. Of course, if $\sigma \geq \tau$, then σ-hyperconnectivity implies τ-hyperconnectivity (as expected, these definitions can be shown to agree with the ones given earlier regarding hyperconnectivity measures).

Axiom (*ii*) of Definition 6.6.3 requires that the σ-levels of a hyperconnectivity pyramid be nested, so that the criterion for hyperconnectivity is increasingly stricter as one goes up the pyramid. In other words, less objects tend to be hyperconnected at large scales than at small scales.

Given a hyperconnectivity class $\mathcal{H}$ on $\mathcal{L}$, furnished with an overlap criterion $\perp$, we can define a simple hyperconnectivity pyramid $\mathbf{H}$ on $\mathcal{L}$, by letting $\mathbf{H}(\sigma) = \mathcal{H}$, for all $\sigma \in \mathbb{Z}$, where the underlying overlap measure is given by the mapping $\perp$ itself. In this case, each $A \in \mathcal{L}$ is either fully hyperconnected, if $A \in \mathcal{H}$, or fully non-hyperconnected, if $A \notin \mathcal{H}$. This supports our earlier observation that hyperconnectivity classes correspond to single-scale hyperconnectivities, where the hyperconnectivity is the same at all scales; i.e., there is no intermediate hyperconnectivity.

Let $\mathcal{L}$ be a lattice, with a fixed sup-generating family $\mathcal{S}$, furnished with an overlap measure $\perp$. Similarly to the multiscale connectivity case, it can be shown fairly easily that the set $\mathcal{M}_H(\mathcal{L}, \perp)$ of all hyperconnectivity measures on $\mathcal{L}$ and the set $\mathcal{Y}_H(\mathcal{L}, \perp)$ of all hyperconnectivity pyramids on $\mathcal{L}$ are complete lattices, under the product partial order (the infimum in those lattices is simply the pointwise infimum). Moreover, we have the following result, which is the hyperconnectivity counterpart of Theorem 6.2.9.

6.6.4 Theorem. Let $\mathcal{L}$ be a lattice with sup-generating family $\mathcal{S}$, furnished with an overlap measure $\perp$. The lattice $\mathcal{M}_H(\mathcal{L}, \perp)$ of hyperconnectivity measures on $\mathcal{L}$ is isomorphic to the lattice $\mathcal{Y}_H(\mathcal{L}, \perp)$ of hyperconnectivity pyramids on $\mathcal{L}$. Moreover, the isomorphism Γ_H from $\mathcal{M}_H(\mathcal{L}, \perp)$ to $\mathcal{Y}_H(\mathcal{L}, \perp)$ is given by

$$\Gamma_H(\varphi_H)(\sigma) = \{A \in \mathcal{L} \mid \varphi_H(A) \geq \sigma\}, \quad \sigma \in \mathbb{Z}, \tag{6.96}$$

with inverse Γ_H^{-1} from $\mathcal{Y}_H(\mathcal{L}, \perp)$ to $\mathcal{M}_H(\mathcal{L}, \perp)$ given by

$$\Gamma_H^{-1}(\mathbf{H})(A) = \bigvee \{\sigma \in \mathbb{Z} \mid A \in \mathbf{H}(\sigma)\}, \quad A \in \mathcal{L}. \tag{6.97}$$

□

The isomorphism between lattices $\mathcal{M}_H(\mathcal{L}, \perp)$ and $\mathcal{Y}_H(\mathcal{L}, \perp)$ is of course a bijection; i.e., to each hyperconnectivity measure φ_H on $\mathcal{L}$, according to the overlap measure $\perp$, there is an associated equivalent hyperconnectivity pyramid $\mathbf{H}$ on $\mathcal{L}$, which consists of the σ-sections of φ_H, also according to the overlap measure $\perp$. Conversely, φ_H can be regenerated by "stacking up" the σ-levels of $\mathbf{H}$. Hence, a multiscale hyperconnectivity on $\mathcal{L}$ can be equivalently specified by either method. Therefore, we say that $\mathcal{L}$ is furnished with a *multiscale hyperconnectivity system* $(\varphi_H, \mathbf{H}) \in \mathcal{M}_H(\mathcal{L}, \perp) \times \mathcal{Y}_H(\mathcal{L}, \perp)$, such that φ_H and $\mathbf{H}$ are equivalent under the bijection given in Theorem 6.6.4. Note that all the definitions regarding full hyperconnectivity, σ-hyperconnectivity and full non-hyperconnectivity, given earlier for hyperconnectivity measures and hyperconnectivity pyramids, agree for a multiscale hyperconnectivity system $(\varphi_H, \mathbf{H})$ on $\mathcal{L}$.

We give below a few examples of multiscale hyperconnectivities (these examples correspond to the multiscale extensions of the hyperconnectivity classes given in Example 4.4.3).

6.6.5 Example.

(a) Let $\mathcal{L} = \mathcal{P}(\mathbb{Z}^n)$, with the points as sup-generators, furnished with a connectivity class $\mathcal{C}$. Let $\{\delta_\sigma(A) = A \oplus \sigma B \mid \sigma \in \mathbb{Z}_+\}$ be a family of dilations on $\mathcal{P}(\mathbb{Z}^n)$, where B contains the origin of $\mathbb{Z}^n$ and $B_v \in \mathcal{C}$, for all $v \in \mathbb{Z}^n$. Consider the overlap measure $\perp$ on $\mathcal{P}(\mathbb{Z}^n)$ given by

$$\perp(\{A_\alpha\}) = \begin{cases} \infty, & \text{if } \bigcap A_\alpha \neq \emptyset \\ -\bigwedge \{\sigma \in \mathbb{Z}_+ \mid \bigcap (A_\alpha \oplus \sigma B) \neq \emptyset\}, & \text{otherwise} \end{cases}. \tag{6.98}$$

Hence, a family is fully overlapping if their intersection is non-empty. Otherwise, it is clear that the more spread apart the sets in the family are, the more negative the degree of overlapping of the family is. It is easy to verify that $(\varphi_H, \mathbf{H})$, where $\mathbf{H} = \mathbf{C}$ in (6.40) and $\varphi_H = \varphi$ in (6.41) (with $\psi_\sigma = \delta_\sigma$, for $\sigma \in \mathbb{Z}_+$), is a multiscale hyperconnectivity system on $\mathcal{P}(\mathbb{Z}^n)$, according to the overlap measure $\perp$ (this example is intended only as an illustration, since $(\varphi_H, \mathbf{H})$ is clearly a dilation-pyramid multiscale connectivity system on $\mathcal{P}(\mathbb{Z}^n)$, there being nothing new here).

(b) (*Graph-Theoretic Degree of Connectivity*). Let $\mathcal{L} = \mathcal{P}(E)$, where $E \subseteq \mathbb{Z}^n$, with the points as sup-generators, furnished with the overlap measure $\perp$ on $\mathcal{P}(E)$ given by

$$\perp(\mathcal{A}) = \begin{cases} \mathrm{Card}(\bigcap \mathcal{A}), & \text{if } \bigcap \mathcal{A} \neq \emptyset \\ -\infty, & \text{otherwise} \end{cases}. \tag{6.99}$$

Therefore, the degree of overlap is the "area of intersection"; i.e., how many points the sets in the family have in common, if the family has a non-empty intersection. Otherwise, the family is fully non-overlapping. Now, let $G = (E, L)$ be a graph. Recall the notions of graph-theoretic k-connectivity in G and degree of connectivity $\kappa_G : \mathcal{P}(E) \to \mathbb{Z}_+$ in G (see Definitions 3.2.9 and 3.2.10, respectively). The mapping $\varphi_H : \mathcal{P}(E) \to \mathbb{Z}$ given by

$$\varphi_H(A) = \begin{cases} \infty, & \text{if } A = O \text{ or } A \in \mathcal{S} \\ \kappa_G(A), & \text{if } A \text{ is connected} \\ -\infty, & \text{otherwise} \end{cases}, \tag{6.100}$$

is a hyperconnectivity measure on $\mathcal{P}(E)$. Denoting by $\mathcal{C}$ the connectivity class of connected sets in G, we can express the associated hyperconnectivity pyramid $\mathbf{H} : \mathbb{Z} \to \mathcal{P}(\mathcal{P}(E))$ by

$$\mathbf{H}(\sigma) = \begin{cases} \{O\} \cup \mathcal{S} \cup \{A \subseteq E \mid A \text{ is } k\text{-connected in } G\}, & \sigma > 0 \\ \mathcal{C}, & \sigma \leq 0 \end{cases}, \quad \sigma \in \mathbb{Z}. \tag{6.101}$$

Hence, the graph-theoretic degree of connectivity and graph-theoretic k-connectivity essentially provide an example of a hyperconnectivity multiscale system. In this framework, hyperconnectivity at scale σ corresponds essentially to σ-connectivity in the graph-theoretical sense, for $\sigma > 0$ (indeed, $\mathbf{H}(\sigma)$ and the σ-overlap criterion $\perp_\sigma$ are the hyperconnectivity class and the overlap criterion of Example 4.4.3(b), respectively, for $\sigma > 0$). Note that this is a "positive" type of connectivity, in the sense employed in Section 6.3.2; i.e., the more positive the degree of connectivity is, the more "strongly connected" the object is.

(c) (*Multiscale Flat Hyperconnectivity*). Let $\mathcal{L} = \text{Fun}(E, \overline{\mathbb{Z}})$, where $E \subseteq \mathbb{Z}^n$, with the pulses as sup-generators. Recall the definition of the threshold operator $X_\tau(f) = \{v \in E \mid f(v) \geq \tau\}$. Consider the overlap measure $\perp$ on $\text{Fun}(E, \overline{\mathbb{Z}})$ given by

$$\perp(\{f_\alpha\}) \;=\; \bigvee \{\sigma \in \mathbb{Z} \mid \textstyle\bigcap_\alpha \{X_\tau(f_\alpha) \mid X_\tau(f_\alpha) \neq \emptyset\} \;\neq\; \emptyset, \forall\, \tau \leq \sigma\}. \tag{6.102}$$

Therefore, the degree of overlap is the maximum "height" below which all the non-empty sections of the individual functions have non-empty intersection. Let $\{\mathcal{C}_\tau \mid \tau \in \mathbb{Z}\}$ be a family of connectivity classes in $\mathcal{P}(E)$. The mapping $\varphi_H \colon \text{Fun}(E, \overline{\mathbb{Z}}) \to \mathbb{Z}$ given by

$$\varphi_H(f) = \bigvee \{\sigma \in \mathbb{Z} \mid X_\tau(f) \in \mathcal{C}_\tau, \forall \tau \leq \sigma\}, \tag{6.103}$$

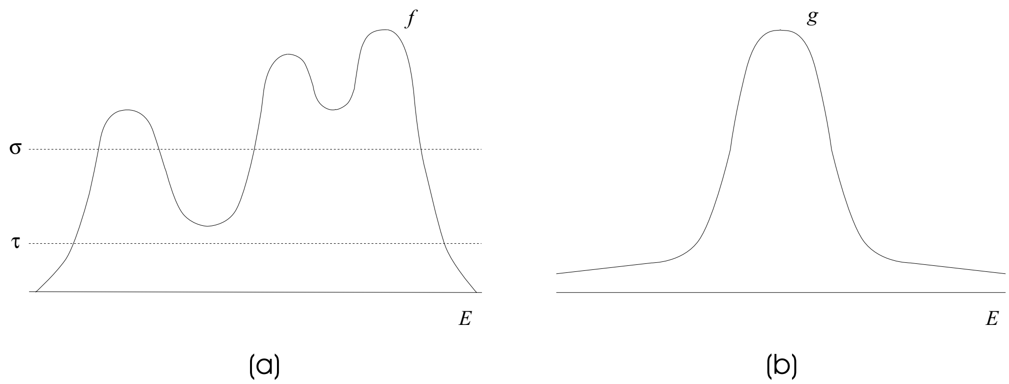


Figure 6.15: Multiscale flat hyperconnectivity example. Here, $E \subseteq \mathbb{Z}$ and the underlying connectivity pyramid reduces to a simple single-scale adjacency connectivity class in $\mathcal{P}(E)$. (a) Function f is τ-connected, but not σ-connected; i.e. $\tau \leq \varphi_H(f) < \sigma$. (b) Function g is fully hyperconnected, or flat hyperconnected; i.e., $\varphi_H(g) = \infty$.

is a hyperconnectivity measure on $\mathrm{Fun}(E, \overline{\mathbb{Z}})$. The associated hyperconnectivity pyramid $\mathbf{H} : \mathbb{Z} \to \mathcal{P}(\mathrm{Fun}(E, \overline{\mathbb{Z}}))$ is given by

$$\mathbf{H}(\sigma) = \{f \in \mathrm{Fun}(E, \overline{\mathbb{Z}}) \mid X_\tau(f) \in \mathcal{C}_\tau, \forall \tau \leq \sigma\}, \quad \sigma \in \mathbb{Z}. \tag{6.104}$$

Hence, the degree of hyperconnectivity in this framework is the maximum "height" below which all the sections of the function are connected; i.e., it is the height of the lowest "disconnecting dip" in the graph of the function. In addition, hyperconnectivity at scale σ essentially corresponds to flat σ-hyperconnectivity, for $\sigma \in \mathbb{Z}$, as defined in Example 4.4.3(c), with $E \subseteq \mathbb{Z}^n$ and $\mathcal{T} = \overline{\mathbb{Z}}$; a function is fully hyperconnected in this framework if it is flat hyperconnected, in the sense of Example 4.4.3(c). See Fig. 6.15 for an illustration (note that the functions in that example are discrete, even though they are represented by continuous curves). $\diamondsuit$

Given a multiscale hyperconnectivity system $(\varphi_H, \mathbf{H})$ on $\mathcal{L}$, the *σ-hyperconnectivity openings* associated with $(\varphi_H, \mathbf{H})$ are given by:

$$\eta_{\sigma,x}(A) = \bigvee\{H \in \mathbf{H}(\sigma) \mid x \leq H \leq A\}, \quad \sigma \in \mathbb{Z},\ x \in \mathcal{S}, \tag{6.105}$$

for $A \in \mathcal{L}$.

Given a $\sigma \in \mathbb{Z}$, a *σ-hyperconnected component* or *hyperconnected σ-grain* of $A \in \mathcal{L}$ is a σ-hyperconnected element $H \in \mathcal{L}$ such that $H \leq A$ and there is no σ-hyperconnected element $H' \in \mathcal{L}$ with $H \leq H' \leq A$. It is clear that, if $x \leq A$ and $\eta_{\sigma,x}(A) \in \mathbf{H}(\sigma)$, then

$\eta_{\sigma,x}(A)$ is the σ-hyperconnected component of $A \in \mathcal{L}$, marked by x; i.e., $\{\eta_{\sigma,x}(A) \mid x \leq A\} \cap \mathbf{H}(\sigma) \subseteq \mathcal{H}_\sigma(A)$, where $\mathcal{H}_\sigma(A)$ denotes the family of hyperconnected σ-grains of A. Unlike the case of multiscale connectivity, it can be shown that the reverse inclusion is not necessarily true (see Section 4.4 for a counterexample in the single-scale case).

As in the single-scale case, σ-hyperconnectivity openings lose some of the nice properties enjoyed by σ-connectivity openings, chiefly among them the ability to uniquely characterize the multiscale hyperconnectivity with which they are associated. However, they do share with σ-connectivity openings the property of forming granulometries parameterized by scale, as given by the next result.

6.6.6 Proposition. Let $\mathcal{L}$ be a lattice with sup-generating family $\mathcal{S}$, furnished with a multiscale hyperconnectivity system $(\varphi_H, \mathbf{H})$. The σ-hyperconnectivity openings associated with $(\varphi_H, \mathbf{H})$, given by (6.105), satisfy the inequality: $\eta_{\sigma,x} \leq \eta_{\tau,x}$, if $\sigma \geq \tau$, for each $x \in \mathcal{S}$; i.e., $\{\eta_{\sigma,x} \mid \sigma \in \mathbb{Z}\}$ constitutes a granulometry on $\mathcal{L}$, for each $x \in \mathcal{S}$. $\square$

Proof. Denoting by $\langle \mathcal{M} \mid \vee \rangle$ the family sup-generated by a family $\mathcal{M}$, it is clear that, if $\sigma \geq \tau$, we have that $\mathrm{Inv}(\gamma_{\sigma,x}) = \langle \mathbf{H}(\sigma) \cap \mathcal{M}^*(x) \mid \vee \rangle \subseteq \langle \mathbf{H}(\tau) \cap \mathcal{M}^*(x) \mid \vee \rangle = \mathrm{Inv}(\gamma_{\tau,x})$, for each $x \in \mathcal{S}$. The desired result then follows from Proposition 2.2.1. Q.E.D.

Given a marker $M \in \mathcal{L}$, the *σ-hyperreconstruction* $\vartheta_\sigma(A \mid M)$ of $A \in \mathcal{L}$ given M is defined by:

$$\vartheta_\sigma(A \mid M) = \bigvee_{x \leq M} \eta_{\sigma,x}(A), \quad \sigma \in \mathbb{Z}. \tag{6.106}$$

Being a supremum of openings, the operator $\vartheta_\sigma(\cdot \mid M)$ is an opening on $\mathcal{L}$, for $\sigma \in \mathbb{Z}$ and a fixed marker $M \in \mathcal{L}$. In addition, it is clear that $\vartheta_\sigma(\cdot \mid M) \leq \vartheta_\tau(\cdot \mid M)$, for $\sigma \geq \tau$; i.e., for each fixed marker $M \in \mathcal{L}$, the family of openings $\{\vartheta_\sigma(\cdot \mid M) \mid \sigma \in \mathbb{Z}\}$ constitutes a granulometry on $\mathcal{L}$, parameterized by scale.

Multiscale hyperconnectivities have the potential to lead to useful image analysis tools, similar to those discussed in Section 6.5 in connection with multiscale connectivities. In particular, the multiscale signal decomposition scheme presented in Section 6.5.1 can be applied here as well. For example, let $\mathcal{L} = \mathrm{Fun}(E, \overline{\mathbb{Z}})$, where $E \subseteq \mathbb{Z}^2$, with the pulses as sup-generators, furnished with the multiscale flat hyperconnectivity system $(\varphi_H, \mathbf{H})$ of Example 6.6.5(c). Given a root marker image $g \in \mathrm{Fun}(E, \overline{\mathbb{Z}})$, consider the multiscale spaces $\{V_\sigma \mid \sigma \in \mathbb{Z}\}$ given by

$$V_\sigma = \mathrm{Inv}[\, \vartheta_\sigma(\cdot \mid g)\,], \quad \sigma \in \mathbb{Z}, \tag{6.107}$$

where $\{\vartheta_\sigma \mid \sigma \in \mathbb{Z}\}$ is the family of σ-hyperreconstruction operators associated with $(\varphi_H, \mathbf{H})$, defined by (6.106). Let the addition and subtraction operations in $\text{Fun}(E, \overline{\mathbb{Z}})$ be given by the usual integer addition and integer subtraction, respectively. This leads to the pyramid transform

$$f \to \{h_{-1}, f_0\} \to \{h_{-1}, h_0, f_1\} \to \cdots \to \{h_{-1}, h_0, \ldots, h_{m-1}, f_m\}, \tag{6.108}$$

for $m \geq 0$, with

$$\begin{cases} f_{j+1} \ = \ \vartheta_{\sigma_0+j+1}(f_j \mid g) \in V_{\sigma_0+j+1} \\ h_j \ = f_j \ - \ f_{j+1} \in \text{Fun}(E, \overline{\mathbb{Z}}) \end{cases}, \tag{6.109}$$

for $j = -1, 0, \ldots, m-1$, where $f_{-1} = f$. Note also that, since σ-reconstruction is an anti-extensive operator, all the detail signals are positive, which can be of great value in practical applications. Since the usual addition and subtraction satisfy the perfect reconstruction condition, the original signal f can be exactly reconstructed from the basic signal f_m and the detail signals $\{h_{-1}, h_0, \ldots, h_{m-1}\}$, by means of the inverse pyramid transform

$$f = f_m \ + \ \textstyle\sum_{j=-1}^{m} h_j. \tag{6.110}$$

Figure 6.16 illustrates the above multiscale signal decomposition scheme using multiscale flat hyperconnectivity, where the underlying family of connectivities reduces to single-scale 4-connectivity. In this example, $\sigma_0 = 100$ and $m = 99$. Only a few of the detail images are shown (most of them are zero). In this example, the root marker image is an impulse located at the global maximum of the image, with height that equals the global maximum value. Note that the basic image f_{99} contains the brightest cells in the image, whereas the low-index detail images contain cells of low brightness (the grayscale range of all detail images was considerably stretched for display purposes). In a sense, this agrees with the remark made earlier that the detail images at lower scale correspond to less relevant information.

The application of multiscale hyperconnectivity to other multiscale image analysis tasks, such as hierarchical segmentation and multiscale features, will be the subject of future research.

6.7 Multiscale Connected Operators

In this section, we show that the notion of connected operators, discussed in Chapter 5, can be extended to the framework of multiscale connectivities. We treat only the continuous multiscale connectivity case; specialization to the discrete case is straightforward.

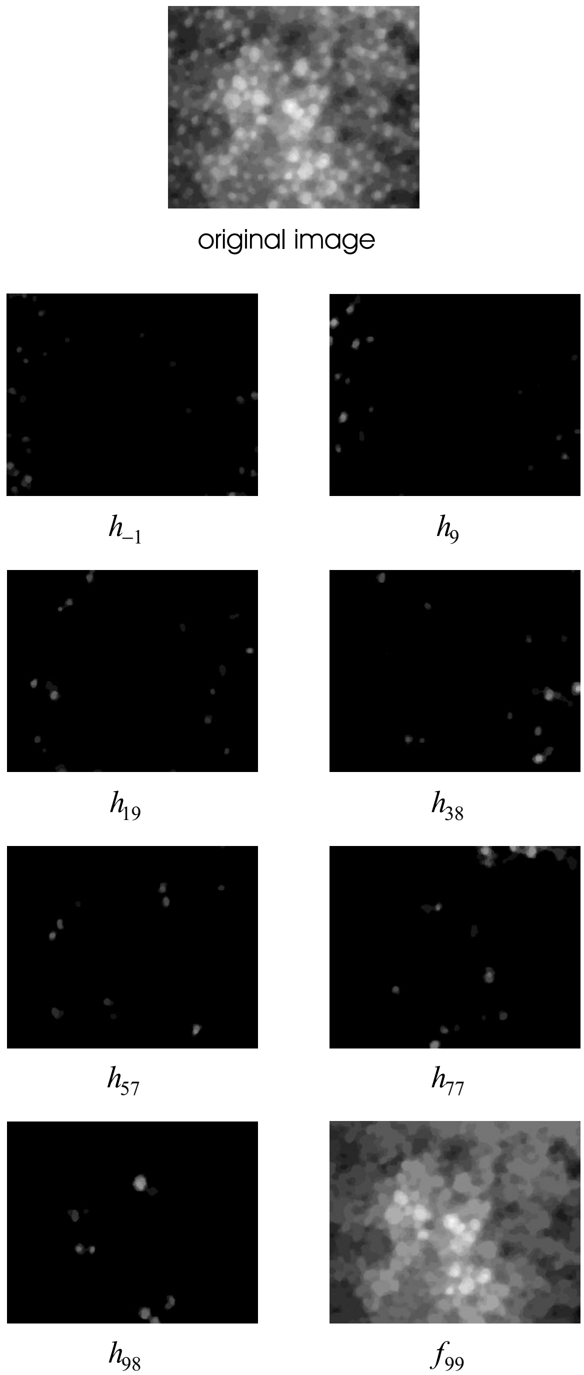


Figure 6.16: Pyramid decomposition of a discrete grayscale image based on a multiscale flat hyperconnectivity. In this example, the root marker image is simply an impulse located at the global maximum of the image, with height that equals the global maximum value.

First, we define the notion of σ-connected operators on function lattices.

6.7.1 Definition. Let $(\varphi, \mathbf{C})$ be a multiscale connectivity system on $\mathcal{P}(E)$. An operator ψ on $\mathrm{Fun}(E, \mathcal{T})$ is said to be σ-connected if it is connected according to $\mathbf{C}(\sigma)$, for $\sigma \in \mathbb{R}$. △

A σ-connected operator is said to be connected at scale σ. An operator is said to be *fully connected* if it is connected at all scales, whereas it is said to be *fully disconnected* if it is not connected at any scale.

Recall from Section 6.5.2 the notion of hierarchical partition of flat zones $\mathbf{z}_f$ of an image $f \in \mathrm{Fun}(E, \mathcal{T})$. The following result is a direct consequence of Definition 5.2.1.

6.7.2 Proposition. An operator ψ on lattice $\mathrm{Fun}(E, \mathcal{T})$ is fully connected if and only if, for every $f \in \mathrm{Fun}(E, \mathcal{T})$, we have that $\mathbf{z}_{\psi(f)}$ is coarser than $\mathbf{z}_f$. □

The following result is a direct consequence of Proposition 5.2.2.

6.7.3 Proposition. If ψ is a σ-connected operator on $\mathrm{Fun}(E, \mathcal{T})$, then ψ is τ-connected, for $\tau \geq \sigma$. □

The above proposition says that if an operator is connected at a certain scale, it is connected at all larger scales. In other words, increasing the connectivity scale preserves the connectedness of the operator. However, lowering the connectivity scale increases clustering of the connected components, which may destroy the connectedness of the operator.

The discussion above suggests the following definition.

6.7.4 Definition. The *degree of connectivity* of an operator ψ on $\mathrm{Fun}(E, \mathcal{T})$ is defined by

$$\Phi(\psi) = -\bigwedge\{\sigma \in \mathbb{R} \mid \psi \text{ is } \sigma\text{-connected}\}. \tag{6.111}$$

△

The degree of connectivity of an operator measures how "insensitive" the operator is to clustering of the flat zones, in the sense that an operator with a large degree of connectivity is connected at low connectivity scales, when the flat zones of the image are larger.

Figure 6.17 illustrates an operator ψ on $\mathcal{P}(\mathbb{Z}^2)$ that acts by filling in pixels in the background that are surrounded by all 4-neighbors in the foreground. This operator is connected only at positive scales, according to the multiscale connectivity of Example 6.1.2,

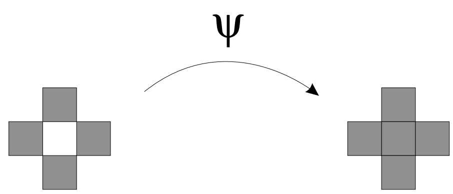


Figure 6.17: The operator ψ exemplified is connected only at positive scales, according to the multiscale connectivity of Example 6.1.2. Hence, $\Phi(\psi) = 0$.

since at negative scales it breaks up the single flat zone corresponding to the background. Accordingly, $\Phi(\psi) = 0$.

Another example is provided by the dilation-pyramid multiscale connectivity of Example 6.3.11. In this case, any connected operator ψ on $\mathcal{P}(E)$ according to the base connectivity class $\mathcal{C}$ is connected at positive scales. For ψ to be σ-connected at negative scales, i.e., for it to have a *positive* degree of connectivity, it must be, to some extent, "insensitive" to the clustering of flat zones introduced by the dilation; i.e., it must treat clusters of grains or pores uniformly (either keep them or remove them). The larger the degree of connectivity $\Phi(\psi)$ is, the larger the clusters of grains or pores that ψ treats uniformly are.

The above discussion on σ-connected operators obviously applies to the binary case as well. In particular, Propositions 5.1.4 and 6.7.3 lead easily to the following result.

6.7.5 Proposition. Let ψ, ϕ, and $\{\psi_i \mid i \in I\}$ be connected operators on $\mathcal{P}(E)$.

(*i*) The operator ψ is σ-connected if and only if the dual operator ψ^* is σ-connected.

(*ii*) If ψ is σ-connected and ϕ is τ-connected, then the composition $\psi\phi$ is $(\sigma \vee \tau)$-connected.

(*iii*) If ψ_i is σ_i-connected, for $i \in I$, then the supremum $\bigvee \psi_i$ and the infimum $\bigwedge \psi_i$ are $(\bigvee \sigma_i)$-connected. □

The ideas discussed in Section 6.2 on grain operators extend directly to the framework of multiscale connectivity.

Given a multiscale connectivity system $(\varphi, \mathbf{C})$ on $\mathcal{P}(E)$, we define *foreground σ-grain operators* and *background σ-grain operators* by

$$\psi_{\sigma,u} = \bigvee_{x \in E} \iota_u \gamma_{\sigma,x} \tag{6.112}$$

$$\phi_{\sigma,v} = \bigwedge_{x \in E} \kappa_v \varphi_{\sigma,x}, \tag{6.113}$$

respectively, where $\{\gamma_{\sigma,x} \mid \sigma \in \mathbb{R}, x \in E\}$ are the σ-connectivity openings associated with $(\varphi, \mathbf{C})$ and $\{\varphi_{\sigma,x} \mid \sigma \in \mathbb{R}, x \in E\}$ are the corresponding *σ-connectivity closings*, given by $\varphi_{\sigma,x} = \gamma^*_{\sigma,x}$, for $\sigma \in \mathbb{R}$, $x \in E$, and ι_u, κ_v are the foreground and background trivial operators, defined by (5.4) and (5.5), respectively. Note that $\psi_{\sigma,u}$ and $\phi_{\sigma,v}$ are obviously σ-connected operators. In addition, $\psi^*_{\sigma,u} = \phi_{\sigma,u}$; i.e., the foreground σ-grain operator and the background σ-grain operator are dual to each other.

Note that, if u is increasing, then $\alpha_{\sigma,u} = \psi_{\sigma,u}$ is an opening on $\mathcal{P}(E)$, called a *σ-grain opening*. Similarly, if v is increasing, then $\beta_{\sigma,v} = \phi_{\sigma,v}$ is a closing on $\mathcal{P}(E)$, called a *σ-grain closing*.

We have the following result.

6.7.6 Proposition.

(a) The σ-grain openings satisfy the inequality: $\alpha_{\sigma,u} \leq \alpha_{\tau,u}$, if $\sigma \geq \tau$; i.e., the family $\{\alpha_{\sigma,u} \mid \sigma \in \mathbb{R}\}$ constitutes a granulometry on $\mathcal{P}(E)$.

(b) The σ-grain closings satisfy the inequality: $\beta_{\sigma,u} \geq \beta_{\tau,u}$, if $\sigma \geq \tau$; i.e., the family $\{\beta_{\sigma,u} \mid \sigma \in \mathbb{R}\}$ constitutes an anti-granulometry on $\mathcal{P}(E)$. □

Proof. (a): If $\sigma \geq \tau$, we have that $\gamma_{\sigma,x} \leq \gamma_{\tau,x} \Rightarrow \iota_u\gamma_{\sigma,x} \leq \iota_u\gamma_{\tau,x} \Rightarrow \alpha_{\sigma,u} = \bigvee \iota_u\gamma_{\sigma,x} \leq \bigvee \iota_u\gamma_{\tau,x} = \alpha_{\tau,u}$, as required.

(b): The proof is completely analogous. Q.E.D.

We remark that, in the literature, granulometries and anti-granulometries based on grain openings and grain closings are indexed by the criteria u and v. On the other hand, the granulometries and anti-granulometries defined in Proposition 6.7.6 are indexed by the scale parameter, which opens up new practical possibilities for the use of σ-grain openings and σ-grain closings in applications.

If the criterion u, or v, is not increasing, then the family $\{\alpha_{\sigma,u} \mid \sigma \in \mathbb{R}\}$, or $\{\beta_{\sigma,v} \mid \sigma \in \mathbb{R}\}$, does not have any regular ordering with respect to the scale parameter σ. However, we can define the following operators:

$$\upsilon_{\sigma,u} = \bigwedge_{\tau \leq \sigma} \psi_{\tau,u} \tag{6.114}$$

$$\omega_{\sigma,v} = \bigvee_{\tau \leq \sigma} \phi_{\tau,v}. \tag{6.115}$$

These operators are clearly σ-connected; furthermore, they are dual to each other: $\upsilon^*_{\sigma,u} = \omega_{\sigma,v}$. In addition, for $\sigma \geq \tau$, we have that $\upsilon_{\sigma,u} \leq \upsilon_{\tau,u}$ and $\omega_{\sigma,v} \geq \omega_{\tau,v}$. Note that these properties resemble those of σ-grain openings and σ-grain closings. As a matter of fact, if u is increasing, then $\upsilon_{\sigma,u} = \psi_{\sigma,u} = \alpha_{\sigma,u}$; similarly, if v is increasing, then $\omega_{\sigma,v} = \phi_{\sigma,u} = \beta_{\sigma,v}$.

Similarly to the single-scale case, we can define a class of operators that simultaneously act on the foreground and the background (recall that $C \lessdot_\sigma A$ means that C is a σ-grain of A):

$$\begin{aligned} \zeta_{\sigma,u,v}(A) &= \bigcup\{C \mid C \lessdot_\sigma A \text{ and } u(C) = 1 \text{ or } C \lessdot_\sigma A^c \text{ and } v(C) = 0\} \\ &= \psi_{\sigma,u}(A) \cup [\phi_{\sigma,v}(A) \setminus A] \;=\; \phi_{\sigma,v}(A) \setminus [A \setminus \psi_{\sigma,u}(A)], \end{aligned} \tag{6.116}$$

for $A \in \mathcal{P}(E)$. These are called *σ-grain operators* and generalize foreground and background σ-grain operators, since $\psi_{\sigma,u} = \zeta_{\sigma,u,1}$ and $\phi_{\sigma,v} = \zeta_{\sigma,1,v}$.

We now have the following characterization.

6.7.7 Proposition. Let ψ be a σ-grain operator on $\mathcal{P}(E)$. For $\tau \leq \sigma$,

$$\psi \text{ is a } \tau\text{-connected operator} \;\Leftrightarrow\; \psi \text{ is a } \tau\text{-grain operator.} \tag{6.117}$$

□

Proof. "⇒": Let $A_1, A_2 \in \mathcal{P}(E)$, and let C be a flat σ-zone (a σ-grain or σ-pore) of both A_1 and A_2. Since, by hypothesis, ψ is τ-connected, it follows from Proposition 5.1.11 that it suffices to show that $\psi(A_1)(C) = \psi(A_2)(C)$. We have that $C = \bigcup C_\alpha$, where C_α are flat σ-zones of A_1 and A_2. But, since ψ is a σ-grain operator, we have that $\psi(A_1)(C_\alpha) = \psi(A_2)(C_\alpha)$, for each α. Hence $\psi(A_1)(C) = \psi(A_2)(C)$, as required.

"⇐": This implication is obvious. Q.E.D.

Hence, lowering the connectivity scale does not destroy the locality (the property of being a grain operator) of a σ-grain operator as long as the operator remains connected. This does not work in the reverse direction: increasing the connectivity scale may destroy locality, even though it preserves connectedness.

Chapter 7

Conclusions

In this dissertation, we have presented a comprehensive theory of connectivity in image processing and analysis. We believe that this dissertation makes a significant contribution to the state of the art on connectivity and its applications in image processing and analysis.

We have provided a thorough review of several existing definitions of connectivity, in ordinary and fuzzy topological spaces or graphs. As far as we know, such an extensive review of traditional concepts of connectivity is lacking in the literature.

Besides reviewing the theory of "morphological connectivity" — the theory of connectivity classes in complete lattices — we have expanded it with new results and new examples, and demonstrated its usefulness with applications based on connected operators.

We have also proposed a novel theoretical framework for the notion of multiscale connectivity, which includes the previous theory of connectivity classes in complete lattices as a special, single-scale case. Among the contributions made by this new theory, we stress the possibility of developing multiscale tools based on connectivity, such as pyramid decompositions, hierarchical segmentation, hierarchical clustering, and multiscale features, which we have demonstrated using simulated and real examples.

Several issues remain to be addressed, which may constitute topics for future research. A few of them are listed in the sequence.

- New examples of connectivities and multiscale connectivities need to be developed, especially examples for grayscale and multispectral images.

- Several interesting theoretical questions are still open. For instance, it would be desirable to find a sufficient condition for a structural closing to be connectivity-preserving, and to be a strong clustering, in the sense of Section 4.3. Similarly, a sufficient condition for the approximation closings of sampling theory to be strong clusterings is needed. As a matter of fact, a *practical* criterion to decide which connectivity-preserving closings are strong clusterings would be quite useful.

- It would be interesting to study how to define hierarchical partitions based on multiscale hyperconnectivities. For instance, this would make possible to use multiscale flat hyperconnectivity, discussed in Section 6.6, in hierarchical segmentation of grayscale images, and to develop multiscale features based on these hierarchical segmentations.
- Most of the examples of connectivity and multiscale connectivity presented here need to be assessed in real applications. In particular, we would like to evaluate flat grayscale connectivity and flat τ-hyperconnectivity, discussed in Sections 4.2 and 4.4, respectively, as segmentation tools for grayscale images.
- Similarly, we would like to investigate the performance of the multiscale tools defined in Section 6.5 in real applications. For instance, we would like to evaluate the use of hierarchical clustering based on multiscale connectivities in real multidimensional unsupervised classification problems. We also would like to consider the use of moments of the clustering curve, or of the clustering spectrum, as a tool for texture classification, and compare the results against other multiscale features, such as granulometric moments or wavelet decomposition coefficients.

Some of the issues listed above are currently under consideration, and we plan to pursue them in our future research.

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